Theorems and Corollaries of Chapter 3

**Theorem 3.1**: Let $R$ and $S$ be rings. Define addition and multiplication on the Cartesian Product $R \times S$ by $(r, s) + (r', s') = (r + r', s + s')$ and $(r, s)(r', s') = (rr', ss')$. Then $R \times S$ is a ring. If both $R$ and $S$ are commutative, then so is $R \times S$. If both $R$ and $S$ have an identity, then so does $R \times S$.

**Theorem 3.2**: Suppose that $R$ is a ring and that $S$ is a subset of $R$ such that

i. $S$ is closed under addition (ie: if $a, b \in S$, then $a + b \in S$);

ii. $S$ is closed under multiplication (ie: if $a, b \in S$, then $ab \in S$);

iii. $0_R$ is in $S$; and

iv. If $a \in S$, then the solution to the equation $a + x = 0_R$ is in $S$.

Then $S$ is a subring of $R$.

**Theorem 3.3**: For any element $a$ in a ring $R$, the equation $a + x = 0_R$ has a unique solution.

**Theorem 3.4**: If $a + b = a + c$ in ring $R$, then $b = c$.

**Theorem 3.5**: For any elements $a$ and $b$ in a ring $R$,

1. $a \cdot 0_R = 0_R = 0_R \cdot a$.
2. $a(-b) = -ab = (-a)b$.
3. $-(-a) = a$.
4. $-(a + b) = (-a) + (-b)$.
5. $-(a - b) = (-a) + b$.
6. $(-a)(-b) = ab$.

If $R$ has an identity, then

7. $(-1_R)a = -a$.

**Theorem 3.6**: Let $S$ be a nonempty subset of a ring $R$ such that

i. $S$ is closed under subtraction (ie: if $a, b \in S$, then $a - b \in S$); and

ii. $S$ is closed under multiplication.

Then $S$ is a subring of $R$. 
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**Theorem 3.7:** Let $R$ be a ring, and let $a$ and $b$ be in $R$. Then the equation $a + x = b$ has the unique solution $x = b - a$.

**Theorem 3.8:** Let $R$ be a ring with identity and $a$ and $b$ elements of $R$. If $a$ is a unit, then each of the equations $ax = b$ and $ya = b$ has a unique solution in $R$.

**Theorem 3.9:** Every field is an integral domain.

**Theorem 3.10:** Cancellation is valid in any integral domain $R$ (i.e., if $a \neq 0_R$ and $ab = ac$ in $R$, then $b = c$).

**Theorem 3.11:** Every finite integral domain is a field.

**Theorem 3.12:** Let $f : R \rightarrow S$ be a homomorphism of rings. Then

1. $f(0_R) = 0_S$.
2. $f(-a) = -f(a)$ for every $a$ in $R$.
3. $f(a - b) = f(a) - f(b)$ for all $a$ and $b$ in $R$.

If $R$ is a ring with identity and $f$ is surjective, then

4. $S$ is a ring with identity, and $f(1_R) = 1_S$.
5. Whenever $u$ is a unit in $R$, then $f(u)$ is a unit in $S$ and $f(u)^{-1} = f(u^{-1})$.

**Corollary 3.13:** If $f : R \rightarrow S$ is a homomorphism of rings, then the image of $f$ is a subring of $S$. 