n-pile Dynamic Nim with Various Endings

by Arthur Holshouser and Harold Reiter

Arthur Holshouser
3600 Bullard St.
Charlotte, NC,
USA, 28208

Harold Reiter
Department of Mathematics
UNC Charlotte
Charlotte, NC 28223

Introduction. In this paper, two players alternate removing a positive number of counters from one of n piles of counters, and the choice of which pile he removes from can change on each move. On his initial move, the player moving first can remove from one pile of his choice at most t counters. On each subsequent move, a player can remove from one pile of his choice at most $f(x)$ counters, where x is the number of counters removed by his opponent on the proceeding move. The game ends when the total number of counters remaining does not exceed k, k being specified in the beginning, and the winner is the player who moves last. We initially studied the strategy for k arbitrary but fixed, $f(x) = x$ and $n = 2$. We then generalized the strategy arising from this $(k, f, n) = (k, x, 2)$ game in a straightforward way to arrive at what we call the "ideal" theorem. This ideal theorem specifies the strategy for an arbitrary triple (k, f, n). Of course, this ideal theorem is not always true, and this led us to pose the problem of finding all triples (k, f, n) for which the conclusion of the ideal theorem **is** true. This paper will give the complete solution to this problem. In [4] we solved the single pile game.

Notation. Z is the set of all integers, Z^+ is the set of positive integers, and

$B = \{1, 2, 4, 8, 16, 32, \cdots\}$ is the set of integer powers of 2.

Definition 1. For all integers $N, \overline{g}(N)$ is the highest power of 2 that divides N, and $\overline{g}(0) = \infty$. Thus $\overline{g}(1) = 1, \overline{g}(-24) = 8$.

Definition 2. Let $m \in Z^+$ be fixed. For all integers $N, \overline{g}_{2^m}(N)$ is the highest power of 2 that divides N, unless $2^m|N$.

If $2^m|N$, then $\overline{g}_{2^m}(N) = \infty$. Thus $\overline{g}_{2^m}(0) = \infty$, $\overline{g}_{2^m}(3 \cdot 2^{m-1}) = 2^{m-1}$, $\overline{g}_{2^m}(-1) = 1$, and $\overline{g}_{2^m}(-5 \cdot 2^{m+1}) = \infty$.

Definition 3. A function $f : Z^+ \to Z^+$ is called *suitable* if it satisfies the two conditions:

1. $\forall N \in Z^+, \overline{g}(N) \leq f(N)$ and
2. $\forall N \in B, f(N) < 2N$.

We will now play our n-pile Dynamic Nim game using any fixed $k \in Z^+ \cup \{0\}$ with any suitable function f. Of course, $f(N) = N, f(N) = 2N - 1$ are two such functions.

Definition 4. If $A_1, A_2, \cdots A_n$ are the n pile sizes, a position can be denoted as $(A_1, A_2, \ldots, A_n)$. For every position $(A_1, A_2, \ldots, A_n)$, we define $g(A_1, A_2, \ldots, A_n)$ to be the smallest winning move size. This means that a winning move is to remove $g(A_1, A_2, \ldots, A_n)$ from one of the piles; however, $g(A_1, A_2, \ldots, A_n)$ by itself would not necessarily tell the player from which piles it is permissible to remove

$g(A_1, A_2, \ldots, A_n)$ from. Also, of course, if $x < g(A_1, A_2, \cdots A_n)$, then the removal of x counters must be a losing move no matter from which of the n piles x counters is removed. Especially note that $g(A_1, A_2, \ldots, A_n) = \infty$ means that all moves are losing moves.

Definition 5. $(Z^+ \cup \{0\}, \oplus)$ is defined as follows. Suppose $U, V \in Z^+ \cup \{0\}$. First, we write U, V in binary, and then we add the two numbers digitwise using the following digit table

$\oplus$	0	1
0	0	1
1	1	0

.

For example, $38 \oplus 21 = 51$ since

$$\begin{array}{rcr} 38 = & & 100110 \\ 21 = & \oplus & 10101 \\ \hline 51 = & & 110011 \end{array} .$$

Of course, $(Z^+ \cup \{0\}, \oplus)$ is an Abelian group. We will now state the "ideal" theorem that we would like to be true for any $(k, f, n), k, n \in Z^+$, where f is suitable. This "ideal" theorem is just a straight forward generalization of the 2-pile strategy for $f(x) = x$. We will deal with $k = 0$ separately.

Ideal Theorem. Let $f : Z^+ \to Z^+$ be suitable and $k \in Z^+$. Using (k, f), we play the n-pile dynamic game. Let us define $m \in Z^+$ by $2^{m-1} \leq k < 2^m$. For every position $(A_1, A_2, \ldots, A_n)$, where $k \leq A_1 + A_2 + \cdots + A_n$, $g(A_1, A_2, \ldots, A_n)$ is defined as the smallest winning move size and is computed by the following rule.

First, define non-negative integers $a_i, \overline{a}_i, i = 1, 2, \cdots, n$, such that $\forall i, 1 \leq i \leq n, A_i = a_i + 2^m \overline{a}_i, 0 \leq a_i < 2^m$. Then $g(A_1, A_2, \ldots, A_n) = g(a_1 + 2^m \overline{a}_1, \cdots, a_n + 2^m \overline{a}_n) = \min(\overline{g}_{2^m}(a_1 + a_2 + \cdots + a_n - k), \overline{g}((2^m \overline{a}_1) \oplus \cdots \oplus (2^m \overline{a}_n)))$. Also, a winning move is to remove $g(A_1, A_2, \ldots, A_n)$ from any pile A_i satisfying $g(A_1, A_2, \ldots, A_n) \leq A_i$.

Remark. From definition 2, it is obvious by the definition that $\overline{g}_{2^m}(a_1 + a_2 + \cdots + a_n - k) \in \{\infty\} \cup \{1, 2, 4, \cdots, 2^{m-1}\}$.

Unfortunately, this "ideal" theorem is not always true because of just one problem. Sometimes the $g(A_1, A_2, \ldots, A_n)$ computed by the theorem is finite, and yet there is no pile size A_i such that $g(A_1, A_2, \ldots, A_n) \leq A_i$.

When $k = 1$, this problem cannot occur no matter what n is. Also, we will show later than when $n \in \{1, 2\}$, this problem cannot occur no matter what k is. Also, if at least one pile size A_i satisfies $2^m \leq A_i$, this problem does not occur.

We now state a companion condition.

Companion Condition. Let k, n be positive integers. Let m be a positive integer satisfying $2^{m-1} \leq k < 2^m$. We say that (k, n) satisfies the *companion condition* if for each position $(a_1, a_2, \cdots, a_n)$ satisfying $(a), (b)$ and (c) below, there is an integer i satisfying $1 \leq i \leq n$ such that $\overline{g}_{2^m}(a_1 + a_2 + \cdots + a_n - k) \leq a_i$.

(a) $0 \leq a_i < 2^m, \forall i, 1 \leq i \leq n,$

(b) $k \leq a_1 + a_2 + \cdots + a_n,$

(c) $\overline{g}_{2^m}(a_1 + a_2 + \cdots + a_n - k) \neq \infty.$

Remark. Later, we will find all (k, n) satisfying the companion condition.

Theorem 1. Suppose $k, n \in Z^+$ and f is suitable. Then the "ideal" theorem is true for (k, f, n) if and only if (k, n) satisfies the companion condition. We also preview Theorem 4.

Theorem 4. If $k, n \in Z^+$ and $f : Z^+ \to Z^+$ are arbitrary, the "ideal" theorem is true for (k, f, n) if and only if (k, n) satisfies the companion condition and f satisfies definition 3.

Proof of Theorem 1. Obviously, the companion condition on (k, n) is a necessary condition. Otherwise, the ideal theorem would not even make sense for some positions.

We will now show that the companion condition on (k, n) is sufficient for the ideal theorem to be true for (k, f, n). The proof is by induction on the total number of counters. First, suppose $A_1 + A_2 + \cdots + A_n = k$. Then $\forall i, 1 \leq i \leq n$, $A_i = A_i + 0 \cdot 2^m$ since $0 \leq A_i \leq k < 2^m$. Since $(A_1, A_2, \cdots, A_n)$ is a terminal position, $g\,(A_1, A_2, \cdots, A_n) = \infty$. Also, by the Theorem, $\text{g}(A_1, A_2, \cdots, A_n) = \min(\overline{g}_{2^m}(0), \overline{g}(0 \oplus \cdots \oplus 0)) = \min(\infty, \infty) = \infty$.

So the induction is started. Let us now consider a position $(A_1, A_2, \cdots, A_n)$, where $k < A_1 + A_2 + \cdots + A_n$. Let us define $(A_1, A_2, \cdots, A_n) = (a_1 + 2^m\overline{a}_1, a_2 + 2^m\overline{a}_2, \cdots, a_n + 2^m\overline{a}_n), \forall i, 1 \leq i \leq n, 0 \leq a_i < 2^m$. The theorem requires $g(a_1 + 2^m\overline{a}_1, \cdots, a_n + 2^m\overline{a}_n) = \min(\overline{g}_{2^m}(a_1 + a_n + \cdots + a_n - k), \overline{g}((2^m\overline{a}_1) \oplus \cdots \oplus (2^m\overline{a}_n)))$.

It is now easy to see from definition 2 that if $2^m | a_1 + a_2 + \cdots + a_n - k$, then $\overline{g}_{2^m}(a_1 + a_2 + \cdots + a_n - k) = \infty$, and the theorem requires $g(a_1 + 2^m\overline{a}_1, \cdots, a_n + 2^m\overline{a}_n) = \overline{g}((2^m\overline{a}_1) \oplus (2^m\overline{a}_2) \oplus \cdots \oplus (2^m\overline{a}_n))$. On the other hand, if 2^m does not divide $a_1 + a_2 + \cdots + a_n - k$, then $\overline{g}_{2^m}(a_1 + a_2 + \cdots + a_n - k) \in \{1, 2, 4, \cdots, 2^{m-1}\}$, and the theorem requires $g(a_1 + 2^m\overline{a}_1, \cdots, a_n + 2^m\overline{a}_n) = \overline{g}_{2^m}(a_1 + a_2 + \cdots + a_n - k)$.

We will consider these two cases separately.

Case 1. $2^m | a_1 + a_2 + \cdots + a_n - k$. We must show that $g(a_1 + 2^m\overline{a}_1, \cdots, a_n + 2^m\overline{a}_n) = \overline{g}((2^m\overline{a}_1) \oplus \cdots \oplus (2^m\overline{a}_n))$. We will consider two subcases of this.

(a) $(2^m\overline{a}_1) \oplus \cdots \oplus (2^m\overline{a}_n) \neq 0$

(b) $(2^m\overline{a}_1) \oplus \cdots \oplus (2^m\overline{a}_n) = 0$.

Subcase a. Define $X = \overline{g}((2^m\overline{a}_1) \oplus \cdots \oplus (2^m\overline{a}_n))$. Now X is the highest power of 2 that divides $(2^m\overline{a}_1) \oplus \cdots \oplus (2^m\overline{a}_n)$. Therefore, $2^m \leq X$ and $X \in B$.

First, we will show that X is a winning move size. Since $A_i = a_i + 2^m\overline{a}_i, i = 1, 2, \cdots, n$, from the definitions of X and $\oplus$, it follows that $X \leq A_i$ for some $1 \leq i \leq n$. By symmetry suppose $X \leq A_1$. We will show that the removal of X from A_1 is a winning move.

Of course, this means that a winning move is to remove X from any pile having this amount in it.

Removing X from A_1 gives a new position $(\overline{A}_1, A_2, \cdots, A_n) = (A_1 - X, A_2, \cdots, A_n)$. Now since $X \in B$ and $2^m \leq X$, we know that $A_1 = a_1 + 2^m\overline{a}_1, \overline{A}_1 = a_1 + 2^m\overline{a}_1^*, 0 \leq a_1 < 2^m$. Note that a_1 is the same in both $A_1$and $\overline{A}_1$. Now if $\overline{A}_1 + A_2 + \cdots + A_n \leq k$, there is nothing to prove. So suppose $\overline{A}_1 + A_2 + \cdots + A_n > k$. Now since $a_1 + a_2 + \cdots + a_n - k$ remains unchanged in the new position $(\overline{A}_1, A_2, \cdots, A_n)$, by induction the Theorem states that $g(\overline{A}_1, A_2, \cdots, A_n) = \overline{g}((2^m\overline{a}_1^*) \oplus \cdots \oplus (2^m\overline{a}_n)) \geq 2X$.

Note that $g(\overline{A}_1, A_2, \cdots, A_n) \geq 2X$ follows from (1) the definition of subtraction in binary, (2) the definitions of X and $\oplus$, (3) the fact the $X \in B$, and (4) the fact that $\overline{g}((2^m\overline{a}_1^*) \oplus \cdots \oplus (2^m\overline{a}_n))$ is the highest power of 2 that divides $(2^m\overline{a}_1^*) \oplus \cdots \oplus (2^m\overline{a}_n)$. This also includes the possibility that $g(\overline{A}_1, A_2, \cdots, A_n) = \infty$. Since $X \in B$ and f is suitable, $f(X) < 2X$. This means the next player, who can remove up to $f(X)$, is confronting a losing position. Next, we show that the removal of Y, where $1 \leq Y < X$, is a losing move no matter from which pile Y is removed from. By symmetry suppose Y is removed from A_1 to give a new pile size $\overline{A}_1 = A_1 - Y$. There is no loss of generality in assuming that $\overline{A}_1 + A_2 + \cdots + A_n \geq k$ since if $0 \leq \overline{A}_1 + A_2 + \cdots + A_n < k$, the moving player could easily adjust his move so that $\overline{A}_1 + A_2 + \cdots + A_n = k$, and still win. Let us now write $Y = y + 2^m\overline{y}, 0 \leq y < 2^m$. Note that $\overline{g}(Y) = \overline{g}(y)$ when $y \neq 0$, and $\overline{g}(Y) = 2^m\overline{g}(\overline{y})$ when $y = 0$. Now $\overline{A}_1 = (a_1 + 2^m\overline{a}_1) - (y + 2^m\overline{y})$. Therefore, $\overline{A}_1 = (a_1 - y) + 2^m(\overline{a}_1 - \overline{y}), 0 \leq a_1 - y < 2^m$ or $\overline{A}_1 = (2^m + a_1 - y) + 2^m(\overline{a}_1 - \overline{y} - 1), 1 \leq 2^m + a_1 - y < 2^m$.

We use the first form of $\overline{A}_1$ when $0 \leq y \leq a_1$ and the second form of $\overline{A}_1$ when $a_1 < y$. Now $(\overline{A}_1, A_2, \cdots, A_n)$ is the new position, and by induction the theorem states that $g(\overline{A}_1, A_2, \cdots, A_n) = \min(\overline{g}_{2^m}(a_1 + a_2 + \cdots + a_n - y - k), \overline{g}((2^m(\overline{a}_1 - \overline{y})) \oplus (2^m\overline{a}_2) \oplus \cdots (2^m\overline{a}_n)))$ or $g(\overline{A}_1, A_2, \cdots, A_n) = \min(\overline{g}_{2^m}(a_1 + a_2 + \cdots + a_n + 2^m - y - k), \overline{g}((2^m(\overline{a}_1 - \overline{y} - 1)) \oplus (2^m\overline{a}_2) \oplus \cdots (2^m\overline{a}_n)))$.

Now $0 \leq y < 2^m$. First, suppose $y \neq 0$. Then since $2^m | a_1 + a_2 + \cdots + a_n - k$, we see that 2^m does not divide $a_1 + a_2 + \cdots + a_n - y - k$ and 2^m does not divide $a_1 + a_2 + \cdots + a_n + 2^m - y - k$. Therefore, if $y \neq 0$, we see by induction that

$g(\overline{A}_1, A_2, \cdots, A_n) = \overline{g}_{2^m}(a_1 + a_2 + \cdots + a_n - y - k) = \overline{g}(y)$ or $g(\overline{A}_1, A_2, \cdots, A_n) = \overline{g}_{2^m}(a_1+a_2+\cdots+a_n+2^m-y-k) = \overline{g}(y)$. Now when $y \neq 0, \overline{g}(Y) = \overline{g}(y+2^m\overline{y}) = \overline{g}(y)$. Therefore, when $y \neq 0, g(\overline{A}_1, A_2, \cdots, A_n) = \overline{g}(Y)$. Since f is suitable, $\overline{g}(Y) \leq f(Y)$. Therefore, $g(\overline{A}_1, A_2, \cdots, A_n) \leq f(Y)$. This means the next player, who can remove up to $f(Y)$, is in a winning position.

Next, suppose $y = 0$. Therefore, $Y = 2^m\overline{y}$. Therefore, $\overline{A}_1 = a_1+2^m(\overline{a}_1-\overline{y})$. Now by induction, the theorem states $g(\overline{A}_1, A_2, \cdots, A_n) = \min(\overline{g}_{2^m}(a_1 + a_2 + \cdots + a_n - k), \overline{g}((2^m(\overline{a}_1-\overline{y}))\oplus(2^m\overline{a}_2)\oplus\cdots\oplus(2^m\overline{a}_n))) = \overline{g}((2^m(\overline{a}_1-\overline{y}))\oplus(2^m\overline{a}_2)\oplus\cdots\oplus(2^m\overline{a}_n))$. This is because $\overline{g}_{2^m}(a_1 + a_2 + \cdots + a_n - k) = \infty$ since $2^m | a_1 + a_2 + \cdots + a_n - k$.

Since $Y = 2^m\overline{y}$ and $Y < X = \overline{g}((2^m\overline{a}_1)\oplus\cdots\oplus(2^m\overline{a}_n))$, we see from the definition of X and from the properties of binary subtraction and from the definition of $\oplus$ that $\overline{g}((2^m(\overline{a}_1 - \overline{y})) \oplus (2^m\overline{a}_2) \oplus \cdots \oplus (2^m\overline{a}_n)) = 2^m\overline{g}(y)$. Therefore, when $y = 0$, we have $Y = 2^m\overline{y}$, which means that $2^m\overline{g}(\overline{y}) = \overline{g}(Y)$. This means that when $y = 0, g(\overline{A}_1, A_2, \cdots, A_n) = 2^m\overline{g}(\overline{y}) = \overline{g}(Y)$.

By definition 3, we know that $\overline{g}(Y) \leq f(Y)$. This means that $g(\overline{A}_1, A_2, \cdots, A_n) \leq f(Y)$. Therefore, the next player, who can remove up to $f(Y)$, is in a winning position.

Subcase b. In subcase b, we have $2^m | a_1 + a_2 + \cdots + a_n - k$ and $(2^m\overline{a}_1) \oplus \cdots \oplus (2^m\overline{a}_n) = 0$. The theorem requires $g(A_1, A_2, \cdots, A_n) = g(a_1 + 2^m\overline{a}_1, a_2 + 2^m\overline{a}_2, \cdots, a_n + 2^m\overline{a}_n) = \min(\overline{g}_{2^m}(a_1 + a_2 + \cdots + a_n - k), \overline{g}((2^m\overline{a}_1) \oplus \cdots \oplus (2^m\overline{a}_n))) = \min(\infty, \infty) = \infty$. This means we must show that all moves are losing moves, and the proof of this is virtually identical to the second part of subcase a.

Before going into case 2, we review the fact that $(A_1, A_2, \cdots, A_n) = (a_1 + 2^m\overline{a}_1, a_2 + 2^m\overline{a}_2, \cdots, a_n + 2^m\overline{a}_n), 0 \leq a_i < 2^m, \forall i, 1 \leq i \leq n$.

Case 2. 2^m does not divide $a_1 + a_2 + \cdots + a_n - k$. As stated previously, from definition 2 this means that $\overline{g}_{2^m}(a_1 + a_2 + \cdots + a_n - k) \in \{1, 2, 4, \cdots, 2^{m-1}\}$. Also, as noted previously, it is easy to see that the theorem requires $g(A_1, A_2, \cdots, A_n) = g(a_1 + 2^m\overline{a}_1, \cdots, a_n + 2^m\overline{a}_n) = \min(\overline{g}_{2^m}(a_1 + a_2 + \cdots + a_n - k), \overline{g}((2^m\overline{a}_1) \oplus \cdots \oplus (2^m\overline{a}_n))) = \overline{g}_{2^m}(a_1 + a_2 + \cdots + a_n - k)$.

Let us now define $x = \overline{g}_{2^m}(a_1+a_2+\cdots+a_n-k)$. Of course, $x \in \{1, 2, 4, \cdots .2^{m-1}\}$. First, we show that x is a winning move size. Now $\forall i, 1 \leq i \leq n, A_i = a_i + 2^m\overline{a}_i, 0 \leq a_i < 2^m$. Now if $0 \leq a_1 + a_2 + \cdots + a_n < k$, then since $k < A_1 + A_2 + \cdots + A_n$, we know that for some $1 \leq i \leq n, \overline{a}_i \neq 0$ must be true. This means that for this $i, x < A_i$ since $x \leq 2^{m-1}$.

Also, if $k \leq a_1+a_2+\cdots+a_n$, then since 2^m does not divide $a_1+a_2+\cdots+a_n-k$, we know by the companion condition that for some $1 \leq i \leq n$ it is true that $\overline{g}_{2^m}(a_1 + a_2 + \cdots + a_n - k) = x \leq a_i$. This means that there will always be at least one pile size A_i such that $x \leq A_i$. By symmetry suppose $x \leq A_1 = a_1 + 2^m\overline{a}_1$. Now $x = x + 0 \cdot 2^m, 1 \leq x < 2^m$, since $x \in \{1, 2, 4, \cdots, 2^{m-1}\}$.

Let us now remove x from A_1 to give a new position $(\overline{A}_1, A_2, \cdots, A_n) = (A_1 - x, A_2, \cdots, A_n)$. We will now show that this is a winning move. Of course, $\forall i, 1 \leq i \leq n$, if $x \leq A_i$, then the removal of x from A_i would also be a winning move. We can assume that $k < \overline{A_1} + A_2 + \cdots + A_n$. Otherwise, there would be nothing to prove since the moving player has already won. Now $(\overline{A}_1, A_2, \cdots, A_n) = (a_1 - x + 2^m\overline{a}_1, a_2 + 2^m\overline{a}_2, \cdots, a_n + 2^m\overline{a}_n), 0 \leq a_1 - x < 2^m, 0 \leq a_i < 2^m, \forall i, 2 \leq i \leq n$ or $(\overline{A}_1, A_2, \cdots, A_n) = (2^m + a_1 - x + 2^m(\overline{a}_1 - 1), a_2 + 2^m\overline{a}_2, \cdots, a_n + 2^m\overline{a}_n), 1 \leq 2^m + a_1 - x < 2^m, 0 \leq a_i < 2^m, \forall i, 2 \leq i \leq n$.

We use the first form when $x \leq a_1$ and the second form when $a_1 < x$. Therefore, by induction, for the new position $(\overline{A}_1, A_2, \cdots, A_n)$ we know that $g(\overline{A}_1, A_2, \cdots, A_n) = \min(\overline{g}_{2^m}(a_1 + a_2 + \cdots a_n - x - k), \overline{g}((2^m\overline{a}_1) \oplus (2^m\overline{a}_2) \oplus \cdots \oplus (2^m\overline{a}_n))$ or $g(\overline{A}_1, A_2, \cdots, A_n) = \min(\overline{g}_{2^m}(a_1 + a_2 + \cdots + a_n + 2^m - x - k), \overline{g}((2^m(\overline{a}_1 - 1)) \oplus (2^m\overline{a}_2) \oplus \cdots \oplus (2^m\overline{a}_n)))$.

Now $x \in \{1, 2, 4, \cdots, 2^{m-1}\}$ and x is the highest power of 2 that divides $a_1 + a_2 + \cdots + a_n - k$.

It is easy to see that the highest power of 2 that divides $a_1 + a_2 + \cdots a_n - x - k$ is no smaller then $2x$. This is true whether $a_1 + a_2 + \cdots a_n - x - k$ is positive, negative, or zero. Also, the highest power of 2 that divides $a_1 + a_2 + \cdots a_n + 2^m - x - k$ is no smaller then $2x$. This is true whether $a_1 + a_2 + \cdots a_n + 2^m - x - k$ is positive, negative, or zero. Now if $2^m | a_1 + a_2 + \cdots a_n - x - k$, we know that $\overline{g}_{2^m}(a_1 + a_2 + \cdots a_n - x - k) = \infty$. Also, if $2^m | a_1 + a_2 + \cdots a_n + 2^m - x - k$, we know that $\overline{g}_{2^m}(a_1 + a_2 + \cdots a_n + 2^m - x - k) = \infty$. Of course ∞ is bigger than any integer. Also, since $x \leq 2^{m-1}$, we know that $2x \leq \overline{g}((2^m\overline{a}_1) \oplus \cdots \oplus (2^m\overline{a}_n))$ and $2x \leq \overline{g}((2^m(\overline{a}_1 - 1)) \oplus 2^m\overline{a}_2 \oplus \cdots \oplus (2^m\overline{a}_n))$.

It is now easy to see that no matter how the details play out we are always going to have $g(\overline{A}_1, A_2, \cdots, A_n) \geq 2x$. Now since $x \in B$, by definition 3, $f(x) < 2x$. Since the next player can remove only up to $f(x)$, this means the next player is confronted with a losing position. This means the moving player made a winning move.

Next, we show that the removal of y, where $1 \leq y < x, x = \overline{g}_{2^m}(a_1 + a_2 + \cdots + a_n - k)$, is a losing move no matter from which pile y is removed. Of course, $1 \leq y < 2^{m-1} < 2^m$ since $x \in \{1, 2, 4, \cdots, 2^{m-1}\}$.

By symmetry suppose y is removed from A_1 to give a new position $(\overline{A}_1, A_2, \cdots, A_n) = (A_1 - y, A_2, \cdots, A_n)$. We will assume that $\overline{A}_1 + A_2 + \cdots A_n \geq k$ since if $0 \leq \overline{A}_1 + A_2 + \cdots A_n < k$, the moving player could easily adjust his move so that $\overline{A}_1 + A_2 + \cdots A_n = k$ and still win.

Now $A_1 = a_1 + 2^m\overline{a}_1, 0 \leq a_1 < 2^m$. Therefore, $(\overline{A}_1, A_2, \cdots, A_n) = (a_1 - y + 2^m\overline{a}_1, a_2 + 2^m\overline{a}_2, \cdots, a_n + 2^m\overline{a}_2), 0 \leq a_1 - y < 2^m, 0 \leq a_i < 2^m, \forall i, 2 \leq i \leq n$

or $(\overline{A}_1, A_2, \cdots, A_n) = (2^m + a_1 - y + 2^m(\overline{a}_1 - 1), a_2 + 2^m\overline{a}_2, \cdots, a_n + 2^m\overline{a}_2), 1 \leq 2^m + a_1 - y < 2^m, 0 \leq a_i < 2^m, \forall i, 2 \leq i \leq n$. We use the first form when $y \leq a_1$ and the second form when $a_1 < y$.

Therefore, by induction for the new position $(\overline{A}_1, A_2, \cdots, A_n)$ we have $g(\overline{A}_1, A_2, \cdots, A_n) = \min(\overline{g}_{2^m}(a_1+a_2+\cdots+a_n-y-k), \overline{g}((2^m\overline{a}_1)\oplus(2^m\overline{a}_2)\oplus\cdots\oplus(2^m\overline{a}_2)))$ or $g(\overline{A}_1, A_2, \cdots, A_n) = \min(\overline{g}_{2^m}(a_1 + a_2 + \cdots + a_n + 2^m - y - k), \overline{g}((2^m(\overline{a}_1 - 1) \oplus (2^m\overline{a}_2) \oplus \cdots \oplus (2^m\overline{a}_n)))$.

Now 2^m does not divide $a_1 + a_2 + \cdots + a_n - k$ by the definition of case 2. Also, x is the highest power of 2 that divides $a_1 + a_2 + \cdots + a_n - k$, and $x \leq 2^{m-1}$. Since $1 \leq y < x, x \in \{1, 2, 4, \cdots, 2^{m-1}\}$, it is easy to see that the highest power of 2 that divides $a_1 + a_2 + \cdots + a_n - y - k$ is $\overline{g}(y)$ and the highest power of 2 that divides $a_1+a_2+\cdots+a_n+2^m-y-k$ is also $\overline{g}(y)$. Also, since $1 \leq y < x, x \in \{1, 2, 4, \cdots, 2^{m-1}\}$, it is obvious that $1 \leq \overline{g}(y) \leq 2^{m-2}$.

From the definitions of $\overline{g}_{2^m}$ and $\overline{g}$, it follows from induction that $g(\overline{A}_1, A_2, \cdots, A_n) = \overline{g}(y)$. Now from definition 3, $\overline{g}(y) \leq f(y)$. Therefore, the next player who can remove up to $f(y)$, is confronted with a winning position. This means the removal of y, where $1 \leq y < x$, is a losing move. ■

Misère. To play the misère game using $k \in Z^+ \cup \{0\}, f$ suitable, let the players play the regular game using $(k+1, f)$. The winner of the regular $(k+1, f)$ will be the winner of the misère (k, f) by agreeing not to undershoot $k + 1$ counters remaining. Let us next take care of the rather easy game that uses $k = 0$ and suitable f. The number of piles, n, is arbitrary.

Theorem 2. For $k = 0$ and f suitable, the strategy for the n-pile game is as follows. For every position $(A_1, A_2, \cdots, A_n)$, $g(A_1, A_2, \cdots, A_n) = \overline{g}(A_1 \oplus A_2 \oplus \cdots \oplus A_n)$, and the winning move is to remove $g(A_1, A_2, \ldots, A_n)$ from any pile A_i satisfying $g(A_1, A_2, \ldots, A_n) \leq A_i$. It is easy to show that such a pile A_i always exists.

Proof. Left to the reader.

When $k = 1$, it is easy to see that for an arbitrary number of piles, $n, (k, n) = (1, n)$ satisfies the companion condition. Also, when $k = 1$, theorem 1 can be cast in the following form. Theorem 3, of course, is also the solution of the misère game $(k, f, n) = (0, f, n)$.

Theorem 3. For $k = 1$ and f suitable, the strategy for the n-pile game is as follows: $\forall$ position $(A_1, A_2, \cdots, A_n), g(A_1, A_2, \cdots, A_n) = \overline{g}(A_1 \oplus A_2 \oplus + \cdots \oplus A_n \oplus 1)$, and a winning move is to remove $g(A_1, A_2, \cdots, A_n)$ from any pile A_i satisfying $g(A_1, A_2, \ldots, A_n) \leq A_i$. Such a pile always exists.

Theorem 4. If $k, n \in Z^+$ and $f : Z^+ \to Z^+$ are all arbitrary, the ideal theorem is true for (k, f, n) if and only if (k, n) satisfies the companion condition, and f is suitable.

Proof. As in theorem 1, it is obvious that the companion condition on (k, n) is necessary since, otherwise, the ideal theorem would not even make sense for some positions.

The only thing that is remaining for us to show is that if (k, n) satisfies the companion condition and if (k, f, n) satisfies the ideal theorem then it is necessary that f be suitable.

Now if (k, f, n) satisfies the ideal theorem, it is obvious that $(k, f, n) = (k, f, 1)$ also must satisfy the ideal theorem since $(A_1, 0, 0, \cdots, 0)$ is also a position in (k, f, n). So we will finish the proof by focusing our attention on single pile games. Now all of the single pile games are essentially the same no matter what k is. For convenience we can just assume that $k = 0$. We first show that f satisfies condition 1 of definition 3. Suppose there exists $x \in Z^+$ such that $f(x) < \overline{g}(x)$. We show that this leads to a contradiction. Consider the position $(2^n\overline{g}(x))$, where $x < 2^n\overline{g}(x)$. Of course, $\overline{g}(x) \in B$ and $2^n\overline{g}(x) \in B$. We can use theorem 2 since we are dealing with a single pile game with $k = 0$.

Now by theorem 2, $g(2^n\overline{g}(x)) = \overline{g}(2^n\overline{g}(x)) = 2^n\overline{g}(x)$. Let us now remove x counters from the pile of $2^n\overline{g}(x)$ counters to get a new position $(2^n\overline{g}(x) - x)$. Now by theorem 2, $g(2^n\overline{g}(x) - x) = \overline{g}(2^n\overline{g}(x) - x) = \overline{g}(x)$. This last step is easy to see since $x < 2^n\overline{g}(x)$ and $2^n\overline{g}(x) \in B$. Also, $f(x) < g(2^n\overline{g}(x) - x) = \overline{g}(x)$ is true by the assumption on x. So removing x is a winning move. But since $x < 2^n\overline{g}(x)$, this means that $g(2^n\overline{g}(x)) = 2^n\overline{g}(x)$ is not true, which contradicts theorem 2 (which is equivalent to contradicting the ideal theorem). Therefore the assumption that there exists $x \in Z^+$ such that $f(x) < \overline{g}(x)$ is false.

Last, we show that f must also satisfy condition 2 of definition 3. Therefore, suppose there exists $x \in B$ such that $f(x) \geq 2x$. Consider the single pile position $(3x)$. Theorem 2 states that $g(3x) = \overline{g}(3x) = \overline{g}(x) = x$. Note that $\overline{g}(3x) = \overline{g}(x) = x$ since $x \in B$.

This means that the removal of x counters from the single pile of $3x$ counters is a winning move.

Since the new pile size is $2x$, this requires $f(x) < g(2x) = \overline{g}(2x) = 2x$, a contradiction since $f(x) \geq 2x$. ∎

Problem. Given (k, n), where $k, n \in Z^+$, we would like to determine whether (k, n) satisfies the companion condition. The following theorems give the complete solution to this problem.

Theorem 5. If $n = 2$, then $(k, n) = (k, 2)$ satisfies the companion condition for all $k \in Z^+$.

Proof. We recall from the companion condition that $2^{m-1} \leq k < 2^m, m \in Z^+$.

Also, (a_1, a_2) is any position satisfying $0 \leq a_1 < 2^m, 0 \leq a_2 < 2^m$, $k \leq a_1 + a_2$ and if 2^m does not divide $a_1 + a_2 - k$. We wish to show that $x \leq a_1$ or $x \leq a_2$, where x is the highest power of 2 that divides $a_1 + a_2 - k$.

First, suppose $a_1 \leq k$. Since $0 < a_1 + a_2 - k$, then $x \leq a_1 + a_2 - k \leq a_2$ is true. Similarly, if $a_2 \leq k$ then $x \leq a_1$.

Let us now suppose that $k < a_1, k < a_2$. Since $2^{m-1} \leq k < 2^m$, we know that $2^{m-1} \leq k < a_1$ and $2^{m-1} \leq k < a_2$. Since x is the highest power of 2 that divides $a_1 + a_2 - k$, and since 2^m does not divide $a_1 + a_2 - k$, we know immediately that $x < 2^m$. This means $x \leq 2^{m-1}$ which means $x \leq 2^{m-1} < a_1$ and $x \leq 2^{m-1} < a_2$. ■

Theorem 6. $(k, n) = (k, 1)$ satisfies the companion condition for all $k \in Z^+$.

Theorem 7. If $k \geq 4, n \geq 4$, then (k, n) does not satisfy the companion condition.

Proof. We need only show that $(k, n) = (k, 4)$ does not satisfy the companion condition when $k \geq 4$.

Since $2^{m-1} \leq k < 2^m, k \geq 4$, it is obvious that $m \geq 3$ must be true. Let us now define positive integers $a_1 \leq a_2 \leq a_3 \leq a_4$ that satisfy the two conditions $a_4 - a_1 \in \{0, 1\}$ and $a_1 + a_2 + a_3 + a_4 = k + 2^{m-1}$. We will now prove the following properties $(a), (b), (c)$ and (d) for the position (a_1, a_2, a_3, a_4). This means the companion condition will not be satisfied for (a_1, a_2, a_3, a_4), We prove

(a) $(\forall i, 1 \leq i \leq 4, 0 \leq a_i < 2^m)$. Now the average of a_1, a_2, a_3, a_4 is $\frac{k}{4} + 2^{m-3} < \frac{2^m}{4} + 2^{m-3} = 3 \cdot 2^{m-3}$. This means $\forall i, 1 \leq i \leq 4, a_i \leq 3 \cdot 2^{m-3} < 2^m$,

(b) $(k \leq a_1 + a_2 + a_3 + a_4)$. This is obvious.

(c) (2^m does not divide $a_1 + a_2 + a_3 + a_4 - k$, and $2^{m-1} | a_1 + a_2 + a_3 + a_4 - k$). Since $a_1 + a_2 + a_3 + a_4 - k = 2^{m-1}$, this is obvious.

(d) $(\forall i, 1 \leq i \leq 4, a_i < 2^{m-1})$. As in (a), $\forall i, 1 \leq i \leq 4, a_i \leq 3 \cdot 2^{m-3}$, which means $a_i < 2^{m-1}$.

Theorem 8. If $k = 2, (k, n) = (2, n)$ satisfies the companion condition if and only if $n \in \{1, 2, 3\}$.

Proof. Since theorems 5 and 6 take care of $n = 1, 2$, we show that $(k, n) = (2, 3)$ satisfies the companion condition. Now $2^{m-1} \leq k = 2 < 2^m, m \in Z^+$, gives $m = 2$. When $n = 3$, the positions are (a_1, a_2, a_3). Now if $\overline{g}_4(a_1 + a_2 + a_3 - 2) \neq \infty$, then $\overline{g}_4(a_1 + a_2 + a_3 - 2) \in \{1, 2\}$. This means that only positions that could possibly cause trouble are $(1, 1, 1), (1, 1, 0), (1, 0, 0)$. Since none of these positions cause any trouble, this part of the proof is complete.

We next show that $(k, n) = (2, 4)$ does not satisfy the companion condition. To do this, consider $(1, 1, 1, 1)$. Now $\overline{g}_4(1 + 1 + 1 + 1 - 2) = 2$, which means we have trouble here. ■

Theorem 9. If $k = 3, (k, n) = (3, n)$ satisfies the companion condition if and only if $n \in \{1, 2, 3, 4)$.

Proof. Now $2^{m-1} \leq k = 3 < 2^m, m \in Z^+$, gives $m = 2$. We will show that $(k, n) = (3, 4)$ satisfies the companion condition. This will also take care of $(k, n) = (3, 3)$ since (a_1, a_2, a_3) is the same as $(a_1, a_2, a_3, 0)$. Now if $\overline{g}_4(a_1+a_2+a_3+a_4-3) \neq \infty$, then $\overline{g}_4(a_1+a_2+a_3+a_4-3) \in \{1, 2\}$. The only positions that could cause trouble are $(1, 1, 1, 1), (1, 1, 1, 0), (1, 1, 0, 0), (1, 0, 0, 0)$, and none of these positions do. To show that $(k, n) = (3, 5)$ does not satisfy the companion condition, consider $(1, 1, 1, 1, 1)$. Now $\overline{g}(1 + 1 + 1 + 1 + 1 - 3) = 2$, which means we have trouble here. This also shows that $(k, n) = (3, n), n \geq 5$, does not satisfy the companion condition. ■

Theorem 10. If $k \geq 4$ is fixed, we can determine whether $(k, n) = (k, 3)$ satisfies the companion condition by applying the following rule. First, define $m \in Z^+$ by $2^{m-1} \leq k < 2^m$. Of course, $m \geq 3$. Then $(k, 3)$ satisfies the companion condition if and only if $k \in \{2^m - 2, 2^m - 1)\}$.

Proof. We must determine necessary and sufficient condition on k so that if (a_1, a_2, a_3) is any position that satisfies

(a) $0 \leq a_i < 2^m$, for all $1 \leq i \leq 3$,

(b) $k \leq a_1 + a_2 + a_3$ and (c) 2^m does not divide $a_1 + a_2 + a_3 - k$, then at least one a_i will satisfy $x \leq a_i$, where x is the highest power of 2 that divides $a_1 + a_2 + a_3 - k$. Of course, $x \in \{1, 2, 4, \cdots, 2^{m-1}\}$.

Let us first suppose (a_1, a_2, a_3) satisfies $a_1 < 2^{m-3}, a_2 < 2^{m-3}, a_3 < 2^{m-3}$. Since $2^{m-1} \leq k$ and the hypothesis of the companion condition requires $k \leq a_1 + a_2 + a_3$, we would have a contradiction since $a_1 + a_2 + a_3 < 3 \cdot 2^{m-3} < 2^{m-1} \leq k$. This means that at least one a_i must satisfy $2^{m-3} \leq a_i$ if the hypothesis of the companion condition is satisfied. This also means we only have to deal with $x \in \{2^{m-2}, 2^{m-1}\}$. This means we must find necessary and sufficient conditions on k so that (a') and (b') are true.

(a') If $a_1 + a_2 + a_3 > k$ and 2^{m-2} is the highest power of 2 that divides $a_1 + a_2 + a_3 - k$, then at least one $a_i \geq 2^{m-2}$.

(b') If $a_1 + a_2 + a_3 > k$ and 2^{m-1} is the highest power of 2 that divides $a_1 + a_2 + a_3 - k$, then at least one $a_i \geq 2^{m-1}$.

(a') Let us assume $a_i < 2^{m-2}, \forall i, 1 \leq i \leq 3$. Since $2^{m-1} \leq k < 2^m$, this means $a_1 + a_2 + a_3 - k < 3 \cdot 2^{m-2} - 2^{m-1} = 2^{m-2}$, a contradiction since $2^{m-2} | a_1 + a_2 + a_3 - k$ would be impossible. This means that (a') places no restrictions on k at all.

(b') Now if (a_1, a_2, a_3) is a position that is compatible with the hypothesis and also at least one $a_i \geq 2^{m-1}$, this position of course, would place no restrictions at all on k. So let us now deal with those positions (a_1, a_2, a_3) that are compatible with the hypothesis and also $a_i < 2^{m-1}, \forall i, 1 \leq i \leq 3$. For these positions, $a_1 + a_2 + a_3 - k <$

$3 \cdot 2^{m-1} - 2^{m-1} = 2^m$. Since (b') assumes that 2^{m-1} is the highest power of 2 that divides $a_1 + a_2 + a_3 - k$, this means that for these positions we must have $a_1 + a_2 + a_3 - k = 2^{m-1}$. Therefore, $a_1 + a_2 + a_3 = k + 2^{m-1}$. First, suppose $2^{m-1} \leq k \leq 2^m - 3$. Then $a_1 + a_2 + a_3 = k + 2^{m-1} \leq 2^m + 2^{m-1} - 3 = 3 \cdot 2^{m-1} - 3$. Note that $(2^{m-1}-1)+(2^{m-1}-1)+(2^{m-1}-1) = 3 \cdot 2^{m-1} - 3$. It is now fairly easy to see that we can find non-negative integers (a_1, a_2, a_3) such that $a_1 \leq a_2 \leq a_3, a_3 - a_1 \in \{0, 1\}$, and $\forall i, 1 \leq i \leq 3, a_i \leq 2^{m-1} - 1 < 2^{m-1}$, and $a_1 + a_2 + a_3 = k + 2^{m-1}$. This means that when $2^{m-1} \leq k \leq 2^m - 3$, the companion condition cannot hold for $(k, n) = (k, 3)$. On the other hand, suppose $2^m - 2 \leq k \leq 2^m - 1$. Then $a_1 + a_2 + a_3 = k + 2^{m-1} \geq 2^m - 2 + 2^{m-1} = 3 \cdot 2^{m-1} - 2$. Suppose now that $\forall i, 1 \leq i \leq 3, a_i \leq 2^{m-1} - 1$. This gives $a_1 + a_2 + a_3 \leq 3 \cdot 2^{m-1} - 3$, a contradiction. This means that when $k \in \{2^m - 2, 2^m - 1\}$ the companion condition must hold for $(k, n) = (k, 3)$. ■

References

[1] Elwyn Berlekamp, John H. Conway, and Richard Guy, *Winning Ways*, Academic Press, Vol. 1, A. K. Peters, 2001.

[2] John H. Conway, *On Games and Numbers*, A. K. Peters, 2001.

[3] Richard K. Guy, Fair Game, 2nd. ed., COMAP, New York, 1989.

[4] Holshouser, A., J. Rudzinski, and H. Reiter, Dynamic One-Pile Nim, *The Fibonacci Quarterly*, vol 41.3, June-July, 2003, pp 253-262.

ams classification: 91A46