Sign Charts and the Test Interval Technique

The purpose of this paper is to elaborate the technique discussed in Math 1100 and Math 1120 for finding the sign chart of a *rational function.* A rational function $r(x)$ is a quotient of two polynomial functions, $p(x)$ and $q(x)$. Of course, if $q(x)$ is the constant function with value 1, then $r(x) = p(x) \div 1 = p(x)$ is a polynomial itself, so all that is said here about rational functions applies to polynomial functions. The sign chart for such a rational function is a depiction of the number line separated into intervals by *branch points.* Plus and minus signs are distributed across the number line depending on the sign of the function at points of the interval.

1. Consider the function $p(x) = (x+4)(x+2)^2(x-2)(x-4)^2$. Note that $p(x)$ is already in *factored form.* The zeros of a polynomial in factored form can be read off without trouble. We have $x = -4, -2, 2$ and 4. The *multiplicities* of -2 and 4 are two. Thus we have four branch points as shown on the chart below.

Note that the four branch points divide the number line into five test intervals, $(-\infty, -4), (-4, -2), (-2, 2), (2, 4), (4, \infty)$. Select a *test point* from each interval. Let's take $-5, -3, 0, 3,$ and 5.

To determine the sign of the function at each test point, build a matrix with test points listed down the side and factors listed along the top. In the current case

test point	$(x+4)$	$(x+2)^2$	$(x-2)$	$(x-4)^2$	$p(x)$
-5	$-$	$+$	$-$	$+$	$+$
-3	$+$	$+$	$-$	$+$	$-$
0	$+$	$+$	$-$	$+$	$-$
3	$+$	$+$	$+$	$+$	$+$
5	$+$	$+$	$+$	$+$	$+$
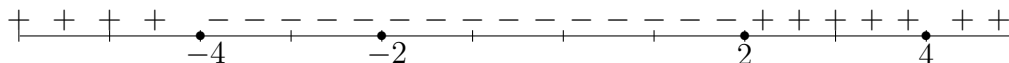
The power of the technique shows up here. It does not matter which point in the interval is selected as the test point. The sign of the function does not change over a test interval. You can see from the sign chart that $p(x)$ changes sign at -4 from positive to negative and at 2 from negative to positive. If the problem we are given is to solve the inequality $p(x) \geq 0$, we could do this

easily at this stage. The solution to $p(x) > 0$ is just $(-\infty, -4) \cup (2, 4) \cup (4, \infty)$. There are four zeros of p to add to this set, so we get $(-\infty, -4] \cup \{-2\} \cup [2, \infty)$.

2. Consider the rational function

$$f(x) = \frac{(x^2 - 4)(2x + 1)}{(3x^2 - 3)(x - 2)}.$$

Notice first that f is not in factored form. Factoring reveals that the numerator and denominator have common factors. Thus

$$f(x) = \frac{(x - 2)(x + 2)(2x + 1)}{3(x - 1)(x + 1)(x - 2)}.$$

We can cancel the common factors with the understanding that we are (very slightly) enlarging the domain of f: $f(x) = \frac{(x+2)(2x+1)}{3(x-1)(x+1)}$. Next find the branch points. These are the points at which f can change signs. Precisely, they are the zeros of the numerator and of the denominator. They are $-2, -1/2, 1, -1$. Again we select test points and find the sign of f at of these points to get the sign chart.
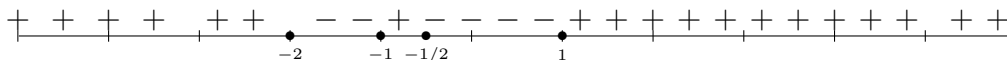

Again suppose that we are solving $f(x) \geq 0$ The solution to $f(x) > 0$ is easy. It is the union of the open intervals with the $+$ signs, $(-\infty, -2) \cup (-1, -1/2) \cup (1, \infty)$. It remains to solve $f(x) = 0$ and attach these solutions to what we have. The zeros of f are -2 and $-1/2$. So the solution to $f(x) \geq 0$ is $(-\infty, -2] \cup (-1, -1/2] \cup (1, \infty)$. Notice that the branch points 1 and -1 are not included since f is not defined at these two points. It has vertical asymptotes at these two places. Technically the value $x = 2$ should not be included in the solution because the function f as originally defined is not defined at $x = 2$. We make this exception repeatedly, however.

3. Find the intervals over which the function $f(x) = (2x-3)^2(x+1)^3$ is increasing. To answer the question, we use Big Theorem A which tells us how to use f' to find the intervals where f is increasing. Specifically, if $f'(x) > 0$ over (a, b), then $f(x)$ is increasing over (a, b). Thus we need the sign chart for $f'(x)$. Use the product rule to find that $f'(x) = 2(2x - 3) \cdot 2 \cdot (x + 1)^3 + 3(x + 1)^2(2x - 3)^2$ and factor this to get $(2x - 3)(x + 1)^2[4(x + 1) + 3(2x - 3)] = (2x - 3)(x + 1)^2(10x - 5)$. The zeros of f' are $3/2, -1$, and $1/2$ and the sign chart is:

$$+ \ + \ + \ + \quad + + + + + + + \underset{-1}{\bullet} + \ + \quad + \underset{1/2}{\bullet} - \ - \underset{3/2}{\bullet} + + + + \ + + + + + + \ + +$$
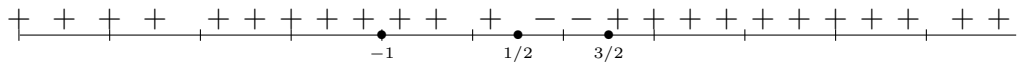
Thus, we can say that f is increasing on $(-\infty, -1)$, $(-1, 1/2)$, and $(3/2, \infty)$. But f is continuous at $x = -1$ so we can amalgamate the first two intervals to get $(-\infty, 1/2)$.

4. Let $g(x) = x^3 - 6x^2 - 15x + 32$. Find the intervals over which g is concave upward. The sign chart for $g''(x) = 6x - 12$ is given below.

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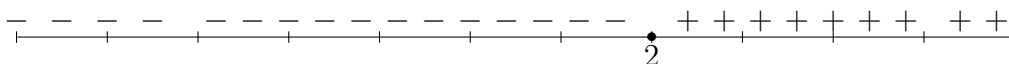
Therefore, by Big Theorem B, g is concave upwards on the interval $(2, \infty)$. The point of inflection is the point on the graph of g where this change takes place. Thus the inflection point is $(2, g(2)) = (2, -14)$.

The next part of this paper is an effort to sort out the types of problems we can solve using sign charts. Suppose we start with a function f. There are three types of behavior we are interested in. There are changing signs, changing direction, and changing concavity. Thus we would like to find

(a) where f changes signs,

(b) where f changes directions, and

(c) where f changes concavity.

In the final example, we will see another problem we can solve using the test interval technique. To answer (1), we need to find the zeros and the vertical asymptotes of f. We do this by constructing the sign chart of f as we did in problems 1 and 2 above. To answer (2), we need to find the relative max and min, and we can do this by constructing the sign chart for f' as we did in problem 3 above. Remember that when f' changes signs, f has either a local max or min. Thus we find the critical points (*stationary and singular*). Using these as branch points, construct the sign chart for f'. To answer (3), we need to find the *inflection points*, ie, the points of the graph of f where f's concavity changes. We do this by constructing the sign chart for f''. When f'' changes from positive to negative, the concavity of f changes from upwards to downwards. And when f'' changes from negative to positive, the concavity of f changes from downwards to upwards.

5. For a final example, recall that $\sqrt{x}$ is a well-defined real number if and only if $x \geq 0$. Use this fact to find the domain of the function $g(x)$ defined by

$$g(x) = \sqrt{(x - 5)(x - 3)(x)(x + 1)^2(x + 4)}.$$

Let $u(x) = (x - 5)(x - 3)(x)(x + 1)^2(x + 4)$. So we need to solve the inequality $u(x) \geq 0$. To this end, note that the roots of $u(x) = 0$ are $x = 5, x = 3, x = 0, x = -1$, and $x = -4$. These branch points split the line into six intervals, $(\infty, -4), (-4, -1), (-1, 0), (0, 3), (3, 5)$, and $(5, \infty)$. Using test points $-5, -2, -1/2, 1, 4$, and 6, we find that $u(-5) > 0, u(-2) < 0, u(-1/2) < 0, u(1) > 0, u(4) < 0, u(6) > 0$. Notice that although $x = -1$ is the endpoint of two abutting intervals over which g is negative, $g(-1) = 0$. Hence we can write the domain of g as $(\infty, -4] \cup [0, 3] \cup [5, \infty) \cup \{-1\}$.