One of the powerful ideas of calculus is the limit concept. The limit concept enables us to discuss the zero over zero problems, which we write as $0/0$. Let's first talk about real number division. We can say that $6/3 = 2$ because $2 \cdot 3 = 6$. Likewise we can say that $0/3 = 0$ because $0 \cdot 3 = 0$. Also, $3/0$ is undefined because, of course there is no real number d such that $d \cdot 0 = 3$.

To see how the $0/0$ problem comes up, begin with two functions f and g which satisfy $f(a) = g(a) = 0$, where a is a point in both their domains. Now we can ask 'what is the behavior of the function $q(x) = \frac{f(x)}{g(x)}$ for x's near a?' Another way to put it is, what is

$$\lim_{x \to a} \frac{f(x)}{g(x)}.$$

The $0/0$ problem is just one of several problems that are called *indeterminant forms*. Other examples of indeterminant forms are $\infty/\infty, \infty - \infty$, and 1^∞. We'll discuss ∞/∞ briefly here as well as $0/0$.

To understand why we want to explore the $0/0$ problem especially, consider the definition of differentiation. Given a function f and a point a in its domain,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists. Notice that if f is continuous (don't be concerned about this term, we'll get to it soon), then $\lim_{h \to 0} f(a+h) - f(a) = 0$. Of course, $\lim_{h \to 0} h = 0$ as well, so here we have the $0/0$ problem.

To handle problems of this type we learn several techniques: factoring, fractional arithmetic, rationalization, and expansion. We'll see examples of each of these. In each case, the method simply allows us to rewrite the quotient $f(x)/g(x)$ in such a way that the $0/0$ problem disappears.

1 Factoring

A very simple example is $f(x) = 2x$ and $g(x) = x$. Then $\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{2x}{x} = \lim_{x \to 0} 2 = 2$. The important idea here is that $\lim_{x \to 0} q(x)$ does not depend on $q(0)$ in any way but only on the values of $q(x)$ for x near 0. Here is a much more interesting example. Find $\lim_{x \to 1} \frac{x^3 + x^2 + 3x - 5}{x^2 - 1}$. Of course we see quickly that we do indeed have a $0/0$ problem. The fact that our numerator $f(x) = x^3 + x^2 + 3x - 5$ has the value 0 when $x = 1$ is important information that enables us to factor it. There is a theorem in algebra (called the Factor Theorem) which tells us if a polynomial like our f has a zero at $x = 1$, then $x - 1$ is a factor of it. In other words, we can write $f(x) = (x - 1)q(x)$ where, in this case, $q(x)$ is a quadratic. Divide $x^3 + x^2 + 3x - 5$ by $x - 1$ to get $x^2 + 2x + 5$, then take the limit of the quotient obtained by eliminating the common factor $x-1$. Thus, we have $\lim_{x\to 1} \frac{x^3+x^2+3x-5}{x^2-1} = \lim_{x\to 1} \frac{(x^2+2x+5)(x-1)}{(x+1)(x-1)} = \lim_{x\to 1} \frac{x^2+2x+5}{x+1} = 8/2 = 4$.

2 Fractional Arithmetic

As an example consider the problem of finding $\lim\limits_{x\to 3} \dfrac{x-3}{\frac{1}{x}-\frac{1}{3}}$ The limit of both the numerator and the denominator is 0, so we must do the fractional arithmetic. The limit becomes

$$\lim_{x\to 3} \frac{x-3}{\frac{3-x}{3x}} = \lim_{x\to 3} \frac{x-3}{-\frac{x-3}{3x}} = \lim_{x\to 3} \frac{1}{-\frac{1}{3x}} = \lim_{x\to 3} -\frac{3x}{1} = -9.$$

Note here, as in the other cases we've seen, we can always create a new problem by flipping the fraction over. If

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L \neq 0,$$

then

$$\lim_{x\to a} \frac{g(x)}{f(x)} = \frac{1}{L}.$$

3 Rationalizing

Consider the problem of finding $\lim\limits_{x\to 5} \dfrac{\sqrt{3x+1}-4}{x-5}$ Again we have the 0/0 problem, and this time we can see that neither factoring nor doing fractional arithmetic can help to resolve the problem. But we can rationalize the numerator to get

$$\lim_{x\to 5} \frac{(\sqrt{3x+1}-4)(\sqrt{3x+1}+4)}{(x-5)(\sqrt{3x+1}+4)} = \lim_{x\to 5} \frac{3(x-5)}{(x-5)(\sqrt{3x+1}+4)} = \frac{3}{4+4} = \frac{3}{8}.$$

Of course, had we started with $\lim_{x\to 5} \frac{x-5}{\sqrt{3x+1}-4}$, we would have rationalized the denominator.

4 Expanding

Next consider the problem

$$\lim_{x\to 0} \frac{(x+1)^3-1}{x}.$$

Some readers will see this as a factoring problem, but most will solve this by expressing $(x+1)^3$ as a polynomial in standard form to get

$$\begin{aligned}\lim_{x\to 0}\frac{(x+1)^3-1}{x} &= \lim_{x\to 0}\frac{x^3+3x^2+3x+1-1}{x}\\ &= \lim_{x\to 0}\frac{x(x^2+3x+3)}{x}\\ &= \lim_{x\to 0} x^2+3x+3 = 3\end{aligned}$$

So now our repertoire includes all four of the methods. Yet there is another method we need to discuss, one which we use to handle the form ∞/∞.

5 ∞/∞

Consider the problem of finding $\lim_{x\to\infty}\frac{(2x^2-3)^2}{(x-1)^4}$. You can see that both the numerator and the denominator are unbounded. The degree of both the numerator and the denominator is 4, so it makes sense to **expand** both. We get the equivalent problem

$$\lim_{x\to\infty}\frac{4x^4-12x^2+9}{x^4-4x^3+6x^2-4x+1}.$$

The method for handling this problem is **division**. At this point in the course, we know how to deal with limit problems like $\lim_{x\to\infty}\frac{1}{x}$ and $\lim_{x\to\infty}\frac{1}{x^2}$. We have reasoned that if the numerator is fixed and the denominator grows without bound, the fraction must have limit zero. Thus, we have

$$\begin{aligned}\lim_{x\to\infty}\frac{(2x^2-3)^2}{(x-1)^4} &= \lim_{x\to\infty}\frac{4x^4-12x^2+9}{x^4-4x^3+6x^2-4x+1}\\ &= \lim_{x\to\infty}\frac{\frac{4x^4-12x^2+9}{x^4}}{\frac{x^4-4x^3+6x^2-4x+1}{x^4}}\\ &= \lim_{x\to\infty}\frac{\frac{4x^4}{x^4}-\frac{12x^2}{x^4}+\frac{9}{x^4}}{\frac{x^4}{x^4}-\frac{4x^3}{x^4}+\frac{6x^2}{x^4}-\frac{4x}{x^4}+\frac{1}{x^4}}\\ &= \frac{\lim_{x\to\infty}\frac{4x^4}{x^4}-\lim_{x\to\infty}\frac{12x^2}{x^4}+\lim_{x\to\infty}\frac{9}{x^4}}{\lim_{x\to\infty}\frac{x^4}{x^4}-\lim_{x\to\infty}\frac{4x^3}{x^4}+\lim_{x\to\infty}\frac{6x^2}{x^4}-\lim_{x\to\infty}\frac{4x}{x^4}+\lim_{x\to\infty}\frac{1}{x^4}}\\ &= \frac{4-0+0}{1-0+0-0+0}=4\end{aligned}$$