Monotonic Arrangements of Undirected Graphs

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1 Abstract

The problem of finding a path in a network each edge of which is at least as long as the previous edge has attracted some attention in recent years. For example, problem 10 of the 2008 American Invitational Mathematics Exam asked a variation of this problem, namely to find the number of paths of maximal length in the 4×4 rectangular grid of dots such that the edge length strictly increases from beginning to end. See [2]. Also, see [1].

A collection $\{a_1, a_2, a_3, \cdots, a_n\}$ of $n \geq 2$ distinct points in the plane is said to be monotone if $a_1, a_2, \cdots, a_n$ can be ordered in some way $a_{i1}, a_{i2}, \cdots, a_{i_n}$ such that the sequence of consecutive distances $D(a_{i1}, a_{i2}), D(a_{i2}, a_{i3}), \cdots, D\left(a_{i_{n-1}}, a_{i_n}\right)$ is non-decreasing. That is, $D(a_{i1}, a_{i2}) \leq D(a_{i2}, a_{i3}) \leq \cdots \leq D\left(a_{i_{n-1}}, a_{i_n}\right)$.

In this note, we show that $n \geq 2$ distinct points in the plane is always monotone if and only if $n = 2, 3$.

We also discuss the same problem for the general abstract graph where the $\binom{n}{2}$ edges of a complete undirected graph on $n \geq 2$ vertices are assigned arbitrary real numbers. We show that this abstract graph can always be monotonically arranged for a binary graph where we assign just two different real numbers $d, D, d < D$, to the $\binom{n}{2}$ edges. We then generalize this binary theorem in a very primitive way to give necessary and sufficient conditions so that any given abstract graph is monotone. The ultimate goal is to create a theorem analogous to Hall's marriage theorem that gives a more reasonable solution to this problem. We then

generalize our results further. This problem was suggested to us by Patrick Vennebush. At the end we give an open ended problem that can be dealt with using generalizations of the ideas in this paper. Key words: binary, trinary, directed, undirected, and complete graphs; Hall's Marriage Theorem. MSC: $05C22$, $05C38$.

2 Introduction

Notation 1 If a, b are points in the plane, then $D(a,b) = D(b,a)$ denotes the distance between a, b.

Definition 1 Suppose $\{a_1, a_2, \cdots, a_n\}, n \geq 2$, is a collection of n distinct points in the plane. A sequence $a_{i1}, a_{i2}, \cdots, a_{i_n}$ where $\{a_1, a_2, \cdots, a_n\} = \{a_{i1}, a_{i2}, \cdots, a_{i_n}\}$ is said to be a monotone path for $a_1, a_2, \cdots, a_n$ if $D(a_{i1}, a_{i2}) \leq D(a_{i2}, a_{i3}) \leq D(a_{i3}, a_{i4}) \leq \cdots \leq D\left(a_{i_{n-1}}, a_{i_n}\right)$.

Definition 2 A collection $\{a_1, a_2, \cdots, a_n\}$ of $n \geq 2$ distinct points in the plane is said to be monotone if there exists a monotone path $a_{i1}, a_{i2}, \cdots, a_{i_n}$ for $\{a_1, a_2, \cdots, a_n\}$.

If no monotone path exists for $\{a_1, a_2, \cdots, a_n\}$, we say that $a_1, a_2, \cdots, a_n$ is not monotone.

In this note, we show that $n \geq 3$ distinct points $a_1, a_2, \cdots, a_n$ in the plane is always monotone if and only if $n = 3$.

In Section 6, we consider the same problem for the abstract graph where the $\binom{n}{2}$ edges of a complete undirected graph on $n \geq 2$ vertices are assigned arbitrary real numbers. We show that this abstract graph can always be monotonically arranged for a binary graph where we assign just two different real numbers $d, D, d < D$, to the $\binom{n}{2}$ edges.

We than generalize this binary theorem in an obvious and primitive way to give necessary and sufficient conditions so that any given abstract graph is monotone. The ultimate goal is to create a theorem analogous to Hall's marriage theorem that gives a more reasonable solution to this problem. In Section 8, we study more general derived graphs. In Section 9 we study trinary graphs which are undirected graphs where each edge is assigned 0 or 1 or 2. We generalize the plane work by showing that a trinary graph on $n = 4$ vertices or $n \geq 6$ vertices is not always monotone.

We have also shown that a trinary graph on $n = 5$ vertices is always monotone, and we give the solution in Section 10. In Section 11, we show that a graph on $n = 5$ vertices is not always monotone if we assign 0 or 1 or 2 or 3 to each of the $\binom{5}{2} = 10$ edges.

3 Cases 3,4,5

Lemma 1 Any three distinct points a_1, a_2, a_3 in the plane is monotone.

Proof. This is obvious. ■

Note 1 We note that n distinct points $a_1 < a_2 < a_3 < \cdots < a_n$ on a straight line is always monotone. To see this, consider $a_1 < a_2 < a_3 < \cdots < a_{2n+1}$ and the ordering $a_{n+1}, a_{n+2}, a_n, a_{n+3}, a_{n-1}, a_{n+4}, a_{n-2}, a_{n+5}, a_{n-3}, \cdots, a_{2n+1}, a_1$.

Also, consider $a_1 < a_2 < \cdots < a_{2n}$ and the ordering $a_n, a_{n+1}, a_{n-1}, a_{n+2}, a_{n-2}, a_{n+3}, a_{n-3}, \cdots, a_{2n}$.

Lemma 2 There exists a nonmonotonal collection a_1, a_2, a_3, a_4 of 4 distinct points in the plane.

Proof. Consider the rectangle $a_1 = (0,0), a_2 = (1,0), a_3 = (0,10), a_4 = (1,10)$.

We show that a_1, a_2, a_3, a_4 is nonmonotonal.

Now each distance $D(a_i, a_j), i \neq j$ must be 1 or 10 or $\sqrt{101}$ where $1 < 10 < \sqrt{101}$.

Suppose $a_{i1}, a_{i2}, a_{i3}, a_{i4}$ is a monotone path for a_1, a_2, a_3, a_4. Now each distance $1, 10, \sqrt{101}$ can be used at most one time in $a_{i1}, a_{i2}, a_{i3}, a_{i4}$ since the adjacent distances in the graph are unequal. Therefore, $D(a_{i1}, a_{i2}) = 1, D(a_{i2}, a_{i3}) = 10, D(a_{i3}, a_{i4}) = \sqrt{101}$ and this is clearly impossible. ■

Note 2 The rectangle in this proof is extended to prove Lemmas 5, 6 and Lemmas 9, 10.

Lemma 3 There exists a nonmonotonal collection a_1, a_2, a_3, a_4, a_5 as of 5 distinct points in the plane.

Proof. Let a_1, a_2, a_3, a_4, a_5 be the 5 vertices, arranged in clockwise order, of a regular pentagon inscribed in a circle.

Let us now move a_1 clockwise on the circle by a very small amount ε and move a_2 counterclockwise on the circle by the same very small amount ε. Thus, a_1 goes to $\overline{a}_1$ and a_2 goes to $\overline{a}_2$. See Fig. 1.

We claim that $\overline{a}_1, \overline{a}_2, a_3, a_4, a_5$ is nonmonotonal.

Now for a_1, a_2, a_3, a_4, a_5, each distance $D(a_i, a_j), i \neq j$, must be d or D where d is the side length of the pentagon and D is the diagonal length.

Therefore, a monotone path $a_{i1}, a_{i2}, a_{i3}, a_{i4}, a_{i5}$ of a_1, a_2, a_3, a_4, a_5 must satisfy $D(a_{i1}, a_{i2}) = D(a_{i2}, a_{i3}) = D(a_{i3}, a_{i4}) = D(a_{i4}, a_{i5}) = d$ or $D(a_{i1}, a_{i2}) = D(a_{i2}, a_{i3}) = D(a_{i3}, a_{i4}) = D(a_{i4}, a_{i5}) = D$. Therefore, a monotone path for a_1, a_2, a_3, a_4, a_5 must consist of consecutive vertices such as a_1, a_2, a_3, a_4, a_5 or alternate vertices such as a_1, a_3, a_5, a_2, a_4. Now all the distances between the vertices of $\overline{a}_1, \overline{a}_2, a_3, a_4, a_5$ are still very close to d or D since ε is very small. Therefore, a monotone path for $\overline{a}_1, \overline{a}_2, a_3, a_4, a_5$, if a monotone path exists, must still consist of consecutive vertices such as $\overline{a}_1, \overline{a}_2, a_3, a_4, a_5$ or alternate vertices such as $\overline{a}_1, a_3, a_5, \overline{a}_2, a_4$. (Sections 6-10 shed much more light on this type of reasoning.)

If we now use symmetry in Fig. 1, we only need to consider the following 12 candidate paths and we see that none of these 12 candidate paths is a monotone path . (We state the contradictions beside of each path.) Note that a chord subtending an arc of $0 \le \theta \le 180°$ has a length of $2R\sin(\theta/2)$.

1. $\overline{a}_1, \overline{a}_2, a_3, a_4, a_5, D(\overline{a}_2, a_3) > D(a_3, a_4)$.

2. $\overline{a}_1, a_5, a_4, a_3, \overline{a}_2, D(\overline{a_1}, a_5) > D(a_5, a_4)$.

3. $\overline{a}_1, a_3, a_5, \overline{a}_2, a_4, D(a_3, a_5) > D(a_5, \overline{a}_2)$.

4. $\overline{a}_1, a_4, \overline{a}_2, a_5, a_3, D(a_4, \overline{a}_2) > D(\overline{a}_2, a_5)$.

5. $a_5, \overline{a}_1, \overline{a}_2, a_3, a_4, D(a_5, \overline{a}_1) > D(\overline{a}_1, \overline{a}_2)$.

6. $a_5, a_4, a_3, \overline{a}_2, \overline{a}_1, D(a_3, \overline{a_2}) > D(\overline{a}_2, \overline{a}_1)$.

7. $a_5, \overline{a}_2, a_4, \overline{a}_1, a_3, D(a_4, \overline{a}_1) > D(\overline{a}_1, a_3)$.

8. $a_5, a_3, \overline{a}_1, a_4, \overline{a}_2, D(a_5, a_3) > D(a_3, \overline{a}_1)$.

9. $a_4, a_3, \overline{a}_2, \overline{a}_1, a_3, D(a_3, \overline{a}_2) > D(\overline{a}_2, \overline{a}_1)$.

10. $a_4, a_5, \overline{a}_1, \overline{a}_2, a_3, D(a_5, \overline{a}_1) > D(\overline{a}_1, \overline{a}_2)$.

11. $a_4, \overline{a}_2, a_5, a_3, \overline{a}_1, D(a_4, \overline{a}_2) > D(\overline{a}_2, a_5)$.

12. $a_4, \overline{a}_1, a_3, a_5, \overline{a}_2, D(a_4, \overline{a}_1) > D(\overline{a}_1, a_3)$.

None of these 12 paths are monotone paths for a_1, a_2, a_3, a_4, a_5 and the contradiction is stated beside of each path. ■

This is studied further in Sections 8-12.

Lemma 4 Let $0 < \varepsilon, 0 < L$.

Define $A \cup B$ by $A = \{a_1, a_2, a_3\}, B = \{b_1, b_2\}$ where $a_1 = (0, L), a_2 = \left(\frac{\varepsilon}{2}, L\right), a_3 = (\varepsilon, L), b_1 = (0, 0), b_2 = (\varepsilon, 0)$.

Then there does not exist for $A \cup B$ a monotone path of the alternating form $x_1, x_2, x_3, x_4, x_5 = a_{i1}, b_{i2}, a_{i3}, b_{i4}, a_{i5}$ where $\{a_{i1}, a_{i3}, a_{i5}\} = \{a_1, a_2, a_3\}$ and $\{b_{i2}, b_{i4}\} = \{b_1, b_2\}$.

The easy proof is left to the reader.

Note 3 We use Lemma 4 in Proof 1 of Lemma 6.

As an interesting exercise, we let the reader prove that $A \cup B$ is monotone for all $0 < \varepsilon, 0 < L$.

4 $2n+2$ Distinct Points in the Plane

We now modify the rectangle of Lemma 2 to prove Lemmas 5, 6 and later to prove Lemmas 9, 10. Although our proofs are different from the proof of Lemma 2, we can see no reason why Lemmas 5, 6 and Lemmas 9, 10 cannot be proved in a way that is very similar to the way that we proved Lemma 2.

We leave this as a project for the reader after the reader has read our proofs.

Lemma 5 There exists a nonmonotonal collection of $2n+2$ distinct points in the plane for each $n \geq 1$. (See also Lemma 9.)

Proof. By Lemma 2, we may assume that $n \geq 2$.

Let $\varepsilon > 0, L > 0$ be fixed.

Define $I\left[0,\varepsilon\right] = \{(x,0) : 0 \leq x \leq \varepsilon\}$ and $I_L\left[0,\varepsilon\right] = \{(x,L) : 0 \leq x \leq \varepsilon\}$.

Define $A = \{a_1, a_2, \cdots, a_n\} \subseteq I_L[0,\varepsilon]$ and $B = \{b_1, b_2, \cdots b_n\} \subseteq I\left[0,\varepsilon\right]$ where $n \geq 2$ and where $a_1, a_2, \cdots, a_n$ are distinct and $b_1, b_2, \cdots b_n$ are distinct. Also, for $0 < \varepsilon < l$, define $\overline{a} = (l, L), \overline{b} = (l, 0)$. Thus, $\overline{a} \notin A, \overline{b} \notin B$.

Let us now agree that $0 < \varepsilon < \frac{l}{100}$ and $0 < l < \frac{L}{100}$.

Thus, we have Fig. 2.

Fig.2. $0 < \varepsilon < \frac{l}{100}, 0 < l < \frac{L}{100}$.

We show that this collection $A \cup \{\overline{a}\} \cup B \cup \left\{\overline{b}\right\}$ of $2n+2$ distinct points in the plane is not monotone.

Define $\overline{A} = A \cup \{\overline{a}\}, \overline{B} = B \cup \left\{\overline{b}\right\}$ and let us denote $\overline{A} = \{\overline{a}_1, \overline{a}_2, \overline{a}_3, \cdots, \overline{a}_{n+1}\}, \overline{B} = \left\{\overline{b}_1, \overline{b}_2, \overline{b}_3, \cdots, \overline{b}_{n+1}\right\}$. Of course, $A = \{a_1, a_2, \cdots, a_n\}, B = \{b_1, b_2, \cdots, b_n\}$.

Now the distance $D(x,y)$ between two distinct points $x, y \in \overline{A}$ is much smaller than the distance $D(\overline{x}, \overline{y})$ between a point $\overline{x} \in \overline{A}$ and a point $\overline{y} \in \overline{B}$.

Also, the distance $D(x,y)$ between two distinct points $x, y \in \overline{B}$ is much smaller than the distance $D(\overline{x}, \overline{y})$ between a point $\overline{x} \in \overline{A}$ and a point $\overline{y} \in \overline{B}$. This is because $0 < l < \frac{L}{100}$ and $D(x,y) \leq l < \frac{L}{100} < L \leq D(\overline{x}, \overline{y})$.

Suppose $x_1, x_2, x_3, \cdots, x_{2n+2}$ is a monotone path for $\overline{A} \cup \overline{B}$. By symmetry we may suppose that $x_1 \in \overline{A}$. From the above distance inequalities, we now show that $x_1, x_2, x_3, \cdots$ must be of the alternating form $x_1, x_2, x_3, \cdots = \overline{a}_{i1}, \overline{b}_{i2}, \overline{a}_{i3}, \overline{b}_{i4}, \overline{a}_{i5}, \overline{b}_{i6}, \cdots, \overline{a}_{i2n+1}, \overline{b}_{i2n+2}$, where $\{\overline{a}_{i1}, \overline{a}_{i3}, \cdots \overline{a}_{i2n+1}\} = \overline{A}$ and $\left\{\overline{b}_{i2}, \overline{b}_{i4}, \cdots, \overline{b}_{i2n+2}\right\} = \overline{B}$. In other words, if $x_1 \in \overline{A}$ then the sequence $x_1, x_2, \cdots, x_{2n+2}$ must alternate between $\overline{A}$ and $\overline{B}$.

To see this, suppose that $x_1, x_2, x_3, \cdots = \overline{a}_{i1}, \overline{a}_{i2}, \overline{a}_{i3}, \cdots, \overline{a}_{ik}, \overline{b}_{ik+1}, \cdots, k \geq 1$. From the above distance inequalities and from the fact that $D(x_1, x_2) \leq D(x_2, x_3) \leq D(x_3, x_4) \leq \cdots$ it is obvious that after we reach $\overline{a}_{ik}, \overline{b}_{ik+1}$ then the sequence $x_1, x_2, x_3, \cdots$ must continue to alternate between $\overline{A}$ and $\overline{B}$. But this is impossible unless the sequence $x_1, x_2, x_3, \cdots$ alternate between $\overline{A}$ and $\overline{B}$ from the very beginning since otherwise we would run out of points in the set $\overline{A}$ since $\overline{A}^{\#} = \overline{B}^{\#}$. In other words, $x_1, x_2, x_3, \cdots$ with $x_1 \in \overline{A}$ must be of the alternating form $x_1, x_2, x_3, \cdots = \overline{a}_{i1}, \overline{b}_{i2,}\overline{a}_{i3}, \overline{b}_{i4}, \overline{a}_{i5}, \overline{b}_{i6}, \cdots$.

We now note that the distance between $\overline{a}$ and $b_i \in B$ is larger than the distance between points $a_j \in A, b_k \in B$. That is, $D(\overline{a}, b_i) > D(a_j, b_k)$.

Also, the distance between $\overline{a}$ and $b_i \in B$ is larger than $D(\overline{a}, \overline{b})$. That is, $D(\overline{a}, b_i) > D(\overline{a}, \overline{b})$.

Also, the distance between $\overline{b}$ and $a_i \in A$ is larger than the distance between points $a_j \in A, b_k \in B$. That is, $D(\overline{b}, a_i) > D(a_j, b_k)$.

Also, the distance between $\overline{b}$ and $a_i \in A$ is larger than $D(\overline{a}, \overline{b})$. That is, $D(\overline{b}, a_i) > D(\overline{a}, \overline{b})$.

We now consider cases 1, 2, 3, 4 for the monotone path $x_1, x_2, x_3, \cdots$ of $A \cup \{\overline{a}\} \cup B \cup \{\overline{b}\} = \overline{A} \cup \overline{B}$ where $x_1 \in \overline{A}$.

Case 1. $x_1 = \overline{a}, x_2 = \overline{b}$.

Case 2. $x_1 = \overline{a}, x_2 = b_{i2} \in B$.

Case 3. $x_1 = a_{i1} \in A, x_2 = \overline{b}$.

Case 4. $x_1 = a_{i1} \in A, x_2 = b_{i2} \in B$.

As always, $a_{ij} \in A$,
$b_{ij} \in B, \overline{a}_{ij} \in \overline{A}, \overline{b}_{ij} \in \overline{B}$.

Case 1 In case 1, we have $x_3 = a_{i3} \in A, x_4 = b_{i4} \in B$ and $x_1, x_2, x_3, x_4, \cdots = \overline{a}, \overline{b}, a_{i3}, b_{i4}, \cdots$ which is impossible since $D(\overline{b}, a_{i3}) > D(a_{i3}, b_{i4})$.

Case 2 In Case 2 we have $x_3 = a_{i3} \in A$ and $x_1, x_2, x_3 = \overline{a}, b_{i2}, a_{i3}$ which is impossible since $D(\overline{a}, b_{i2}) > D(b_{i2}, a_{i3})$.

Case 3 In Lemma 5, we are assuming that $n \geq 2$. Now since $D(a_{i1}, \overline{b}) > D(\overline{b}, \overline{a})$ we see that $x_3 \neq \overline{a}$. Therefore, $x_3 = a_{i3} \in A, x_4 = b_{i4} \in B$ and we have $x_1, x_2, x_3, x_4 \cdots = a_{i1}, \overline{b}, a_{i3}, b_{i4}, \cdots$ which is impossible since $D(\overline{b}, a_{i3}) > D(a_{i3}, b_{i4})$.

Case 4 For Case 4, we consider subcases (a), (b) for the monotone path $x_1, x_2, x_3, \cdots$.

(a) $x_1, x_2, x_3, \cdots = a_{i1}, b_{i2}, a_{i3}, b_{i4}, \cdots, x_{2k+1} = \overline{a}, \cdots, \overline{b}, \cdots, k \geq 1$, and

(b) $x_1, x_2, x_3, \cdots = a_{i1}, b_{i2}, a_{i3}, b_{i4}, \cdots, x_{2k} = \overline{b}, \cdots, \overline{a}, \cdots, k \geq 2$.

By symmetry in the reasoning, we can assume Case (a) which means that the monotone path $x_1, x_2, x_3, \cdots$ gets to $\overline{a}$ before it gets to $\overline{b}$. The reasoning for Case (b) is the same as Case (a).

Now in Case (a) we see that $x_{2k+2} \neq \overline{b}$ since $D(b_{i2k}, \overline{a}) > D(\overline{a}, \overline{b})$.

Therefore, since $x_1, x_2, x_3, \cdots$ is assumed to be a monotone path, we know that $x_{2k+2} = b_{i,2k+2} \in B$ and $x_{2k+3} = a_{i,2k+3} \in A$ and this is impossible since $D(\overline{a}, b_{i,2k+2}) > D(b_{i.2k+2}, a_{i,2k+3})$.

Therefore, $x_1, x_2, x_3, \cdots$ cannot be a monotone path which completes the proof of Lemma 5. ■

5 $2n+3$ District Points in the Plane

Lemma 6 There exists a nonmonotonal collection of $2n+3$ district points in the plane for each $n \geq 1$. (See also Lemma 10). **Proof 1.** By Lemma 3, we may assume that $n \geq 2$.

Let $\varepsilon > 0, L > 0$ be fixed. As in Lemma 5, define $I[0,\varepsilon] = \{(x,0) : 0 \leq x \leq \varepsilon\}$ and $I_L[0,\varepsilon] = \{(x,L) : 0 \leq x \leq \varepsilon\}$.

Define $A = \{a_1, a_2, \cdots, a_{n+1}\} \subseteq I_L[0,\varepsilon]$ and $B = \{b_1, b_2, \cdots, b_n\} \subseteq I(0,\varepsilon)$ where $n \geq 2$ and where $a_1, a_2, \cdots, a_{n+1}$ are distinct and $b_1, b_2, \cdots, b_n$ are distinct.

Also, for $0 < \varepsilon < l$ define $\overline{a} = (l, L), \overline{b} = (l, 0)$. Therefore, $\overline{a} \notin A, \overline{b} \notin B$.

As in Lemma 5, let us now agree that $0 < \varepsilon < \frac{l}{100}$ and $0 < l < \frac{L}{100}$.

Thus, we again have Fig. 2.

From Lemma 4 and by the mathematical inductive proof that we now use, we may assume for all $n \geq 2$ that the set $A \cup B$ does not have a monotone path of the alternating form $x_1, x_2, x_3, \cdots, x_{2n+1} = a_{i1}, b_{i2}, a_{i3}, b_{i4}, a_{i5}, b_{i6}, \cdots, a_{i,2n+1}$. That is, we assume that no monotone path $x_1, x_2, \cdots, x_{2n+1}$ for $A \cup B$ exists with $\{x_1, x_3, x_5, \cdots, x_{2n+1}\} = A$ and $\{x_2, x_4, x_6, \cdots, x_{2n}\} = B$. We do not need an assumption this strong; however, when $n \geq 3$, by the inductive proof that we use, we may assume that $A \cup B$ is not monotone at all. This means for $n \geq 3$ that $A \cup B$ does not have any type of monotone path $x_1, x_3, x_5, \cdots, x_{2n+1}$. We do not need this assumption about $A \cup B$ in proof 2.

We note that in Lemma 4 and later in the inductive proof that we now use the bigger that we make L the better off the proof is. In other words, there is no conflict in making L bigger since we only require $0 < \varepsilon < \frac{l}{100}$ and $0 < l < \frac{L}{100}$. The fact that we can enlarge L without any conflict allows us to use induction to guarantee that $A \cup B$ does not have any type of monotone path. We now show that the collection $A \cup \{\overline{a}\} \cup B \cup \{\overline{b}\}$ of $2n+3, n \geq 2$, distinct points in the plane is not monotone.

Again, define $\overline{A} = A \cup \{\overline{a}\}, \overline{B} = B \cup \{\overline{b}\}$. Let us now call $\overline{A} = \{\overline{a}_1, \overline{a}_2, \overline{a}_3, \cdots, \overline{a}_{n+2}\}$ and $\overline{B} = \{\overline{b_1}, \overline{b}_2, \overline{b}_3, \cdots, \overline{b}_{n+1}\}$. Of course, $A = \{a_1, a_2, \cdots, a_{n+1}\}, B = \{b_1, b_2, \cdots, b_n\}$.

Using the same type of distance argument as in Lemma 5 and using the fact that $\overline{A}^{\#} = n+2, \overline{B}^{\#} = n+1$, we see that a monotone path $x_1, x_2, \cdots, x_{2n+3}$ of $\overline{A} \cup \overline{B}$ must be one of the two forms (a), (b).

(a) $x_1, x_2, x_3, \cdots = \overline{a}_{i1}, \overline{a}_{i2}, \overline{b}_{i3}, \overline{a}_{i4}, \overline{b}_{i5}, \overline{a}_{i6}, \cdots, \overline{b}_{i,2n+3}$.

(b) $x_1, x_2, x_3 = \overline{a}_{i1}, \overline{b}_{i2}, \overline{a}_{i3}, \overline{b}_{i4}, \cdots, \overline{b}_{i,2n+2}, \overline{a}_{1,2n+3}$.

In (a), $\overline{a}_{i2}, \overline{b}_{i3}, \overline{a}_{i4}, \overline{b}_{i5}, \cdots$ alternates between $\overline{A}$ and $\overline{B}$ and in (b) $\overline{a}_{i1}, \overline{b}_{i2}, \overline{a}_{i3}, \overline{b}_{i4}, \cdots$ alternates between $\overline{A}$ and $\overline{B}$.

We have 7 cases to consider. Cases 1, 2, 3 are cases for (a) and cases 4, 5, 6, 7 are cases for (b).

Case 1. $x_1 = \overline{a}, x_2 = a_{i2} \in A$.

Case 2. $x_1 = a_{i1} \in A, x_2 = \overline{a}$.

Case 3. $x_1 = a_{i1} \in A, x_2 = a_{i2} \in A$.

Case 4. $x_1 = \overline{a}, x_2 = \overline{b}$.

Case 5. $x_1 = \overline{a}, x_2 = b_{i2} \in B$.

Case 6. $x_1 = a_{i1} \in A, x_2 = \overline{b}$.

Case 7. $x_1 = a_{i1} \in A, x_2 = b_{i2} \in B$.

If we study the proof of Lemma 5, we see that Cases 2, 3, 6, 7 are proved in the proof of Lemma 5 since we know that in cases 2, 3, 6, 7 the subsequence $x_2, x_3, x_4, x_5, \cdots$ cannot be a monotone path for $\left[A \cup \{\overline{a}\} \cup B \cup \left\{\overline{b}\right\}\right] \setminus \{x_1\}$ since$\left[A \cup \{\overline{a}\} \cup B \cup \left\{\overline{b}\right\}\right] \setminus \{x_1\}$ is not monotone in Cases 2, 3, 6, 7. Therefore we only need to consider Cases 1, 4, 5.

Case 1. $x_1 = \overline{a}, x_2 = a_{i2} \in A$.

We consider two subcases (a), (b) of Case 1.

(a) $x_1, x_2, x_3, \cdots = \overline{a}, a_{i2}, b_{i3}, a_{i4} \cdots, a_{i2k}, \overline{b}, a_{i,2k+2}, b_{i,2n+3}, \cdots$ where $1 \leq k \leq n$.

(b) $x_1, x_2, x_3, \cdots = \overline{a}, a_{i2}, b_{i3}, a_{i4}, b_{i5}, a_{i6}, \cdots, a_{i,2n+2}, \overline{b}$.

In subcase (a), $\overline{b}$ is not the last term in the monotone path $x_1, x_2, x_3, \cdots$ and in subcase (b), $\overline{b}$ is the last term in the monotone path $x_1, x_2, x_3, \cdots$.

Now subcase (a) is impossible since $D\left(\overline{b}, a_{i,2k+2}\right) > D\left(a_{i,2k+2}, b_{i,2k+3}\right)$.

Also, subcase (b) is impossible because the subsequence $a_{i2}, b_{i3}, a_{i4}, b_{i5}, \cdots, a_{i,2n+2}$ tells us that the set $A \cup B$ has a monotone path $a_{i2}, b_{i3}, a_{i4}, b_{i5}, \cdots$ that starts at $a_{i2} \in A$ and alternates between set A and set B and this contradicts the inductive assumption that we make about $A \cup B$. Therefore, Case 1 is impossible.

Case 4. Now, $x_1, x_2, x_3 \cdots = \overline{a}, \overline{b}, a_{i3}, b_{i4}, \cdots$ which is impossible since $D\left(\overline{b}, a_{i3}\right) > D\left(a_{i3}, b_{i4}\right)$.

Case 5. Now $x_1, x_2, x_3, \cdots = \overline{a}, b_{i2}, a_{i3}, \cdots$ which is impossible since $D\left(\overline{a}, b_{i2}\right) > D\left(b_{i2}, a_{i3}\right)$.

This completes the proof of Lemma 6. ■

Lemma 6 can also be proved by the drawing of Proof 2, when $n \geq 2$.

Proof 2. The following drawing can be used to prove Lemma 6 when $n \geq 2$.

We use $A = \{a_1, a_2, \cdots, a_n\} \subseteq I_L[0, \varepsilon]$ and $B = [b_1, b_2, \cdots, b_n] \subseteq I[0, \varepsilon]$ where $a_1, a_2, \cdots, a_n$ are distinct and $b_1, b_2, \cdots, b_n$ are distinct.

Let $\overline{a} = (l, L), a^* = (.99l, L), \overline{b} = (l, 0)$ where $0 < \varepsilon < \frac{l}{100}, 0 < l < \frac{L}{100}$.

We do not need to assume anything about the monotonicity of $A \cup B$ in Proof 2.

We now use a proof that is similar to Proof 1 to show that $A \cup \{\overline{a}, a^*\} \cup B \cup \{\overline{b}\}$ is not monotone.

We first define $\overline{A} = A \cup \{\overline{a}, a^*\}, \overline{B} = B \cup \{\overline{b}\}$.

We show that a monotone path $x_1, x_2, x_3, \cdots$ of $\overline{A} \cup \overline{B}$ must alternate between $\overline{A}$ and $\overline{B}$ exactly as in Proof 1. As in Proof 1, we consider all possible cases for x_1, x_2 where $x_1 \in \overline{A}, x_2 \in \overline{A}$ or $x_1 \in \overline{A}, x_2 \in \overline{B}$.

These cases are as follows.

1. $x_1 \in \{\overline{a}, a^*\}, x_2 \in \{\overline{a}, a^*\}$.

2. $x_1 \in \{\overline{a}, a^*\}, x_2 \in A$.

3. $x_1 \in A, x_2 \in \{\overline{a}, a^*\}$.

4. $x_1 \in A, x_2 \in A$.

5. $x_1 \in \{\overline{a}, a^*\}, x_2 \in B$

6. $x_1 \in \{\overline{a}, a^*\}, x_2 = \overline{b}$.

7. $x_1 \in A, x_2 = \overline{b}$

8. $x_1 \in A, x_2 \in B$

Cases 1, 2, 5, 6 are taken care of in the proof of Lemma 5.

We now leave the details of Cases 3, 4, 7, 8 as an exercise for the reader. These details are slightly more complex than in Proof 1. ■

As stated previously, we can see no reason why the ideas used in the proof of Lemma 2 cannot also be used to prove Lemmas 5, 6.

Also, the drawing used in our proofs of Lemmas 5, 6 can be extended further by placing more points on the upper right and lower right.

6 Abstract Graphs

Let $a_1, a_2, a_3, \cdots, a_n, n \geq 2$, be n distinct vertices and let us connect an undirected edge $x_{ij}, i \neq j$, between each pair $\{a_i, a_j\}$ of distinct vertices. Let us now assign to each edge $x_{ij}, i \neq j$, an arbitrary real number a_{ij}.

We now call $D(a_i, a_j) = D(a_j, a_i) = a_{ij}, i \neq j$, the distance between vertex a_i and vertex a_j. We do not require $D(a, b) + D(b, c) \geq D(a, c)$.

Definition 1 and Definition 2 of Section 2 remain unchanged for these abstract graphs.

Definition 3 A binary undirected graph on n vertices is an undirected graph in which each $a_{ij} \in \{d, D\}$ where $d < D$. We can also let $a_{ij} \in \{0, 1\}$. Of course this is equivalent to defining an undirected graph where an edge is called 1 and the absence of an edge is called 0.

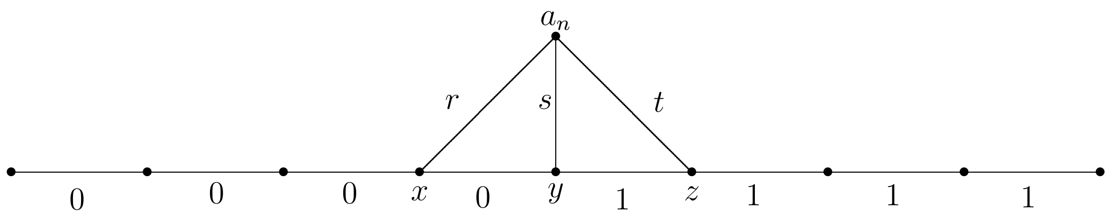
Lemma 7 A binary graph on $n \geq 2$ vertices is always monotone.

Proof. We use mathematical induction on n. Lemma 7 is obviously true for $n = 2$. Therefore, let $n \geq 3$ and consider a binary graph on $\{a_1, a_2, \cdots, a_n\}$. By induction, the binary subgraph on $\{a_1, a_2, \cdots, a_{n-1}\}$ has a monotone path $x_1, x_2, x_3, \cdots, x_{n-1}$.

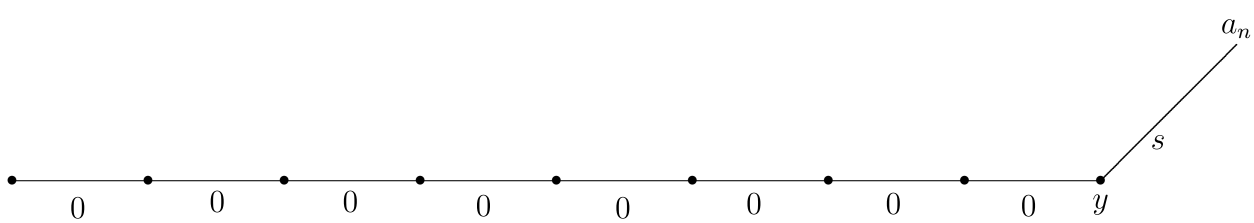
We show how to add vertex a_n to this monotone path $x_1, x_2, \cdots, x_{n-1}$ to get a monotone path for $\{a_1, a_2, \cdots, a_n\}$.

Now the monotone path $x_1, x_2, \cdots, x_{n-1}$ for $\{a_1, a_2, \cdots, a_{n-1}\}$ must be one of three types (a), (b), (c). We also show a_n in the 3 drawings.

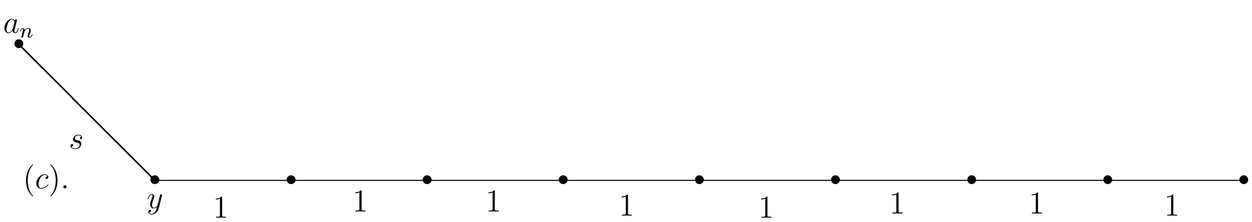
(a).



(b).



(c).



Type (a) Now (r, s, t) can have 8 different values since $r, s, t \in \{0, 1\}$.

1. $(r, s, t) = (0, 0, 0)$. Add x, a_n, y and take out edge xy or add y, a_n, z and take out edge yz to get a monotone path for $\{a_1, a_2, \cdots, a_n\}$.

2. $(r, s, t) = (0, 0, 1)$. Add x, a_n, y and take out edge xy or add y, a_n, z and take out edge yz.

3. $(r, s, t) = (0, 1, 0)$. Add x, a_n, y and take out edge xy.

4. $(r,s,t) = (0,1,1)$. Add x, a_n, y and take out edge xy or add y, a_n, z and take out edge yz.

5. $(r,s,t) = (1,0,0)$. Add y, a_n, z and take out edge yz.

6. $(r,s,t) = (1,0,1)$. Add y, a_n, z and take out edge yz.

7. $(r,s,t) = (1,1,0)$. Add x, a_n, y and take out edge xy.

8. $(r,s,t) = (1,1,1)$. Add x, a_n, y and take out edge xy or add y, a_n, z and take out edge yz.

Type (b) Add y, a_n to get a monotone path for $\{a_1, a_2, \cdots, a_n\}$.

Type (c) Add a_n, y to get a monotone path for $\{a_1, a_2, \cdots, a_n\}$. ■

Observation 1 The proof of Lemma 2 tells us immediately that the conclusion of Lemma 7 is false if we allow the a_{ij} of an abstract graph to have three different values, say $a_{ij} \in \{0, 1, 2\}$. We deal further with $a_{ij} \in \{0, 1, 2\}$ in Lemmas 9, 10.

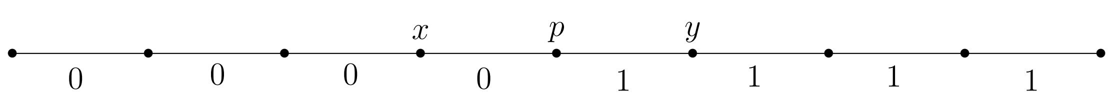
A Practical Construction 1 A practical construction for Lemma 7 is to use the proof of Lemma 7 step by step with each of the subgraphs 1. $\{a_1, a_2\}$, 2.$\{a_1, a_2, a_3\}$, 3.$\{a_1, a_2, a_3, a_4\}, \cdots, (n-1)$. $\{a_1, a_2, a_3, \ldots, a_n\}$.

We add a_3 to graph 1, add a_4 to graph 2, add a_5 to graph 3, $\cdots$, to get monotone paths on $\{a_1, a_2, a_3\}, \{a_1, a_2, a_3, a_4\}, \{a_1, a_2, a_3, a_4, a_5\}, \cdots, \{a_1, a_2, a_3, \cdots, a_n\}$.

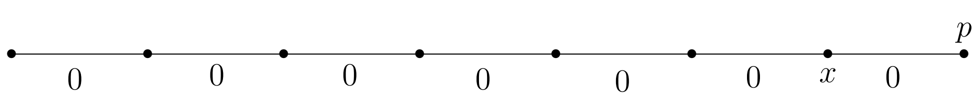
Lemma 8 Any monotone path $x_1, x_2, x_3, \cdots, x_n$ of a binary graph on n vertices $\{a_1, a_2, \cdots, a_n\}$ can be constructed by the above Practical Construction.

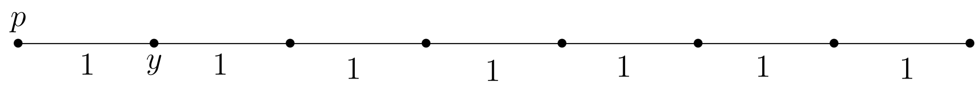
Proof. We consider cases (a), (b), (c) for an arbitrary monotone path $x_1, x_2, x_3, \cdots, x_n$ of a binary graph on $\{a_1, a_2, \cdots, a_n\}$. Cases (b), (c) are left as easy exercises for the reader.

(a)



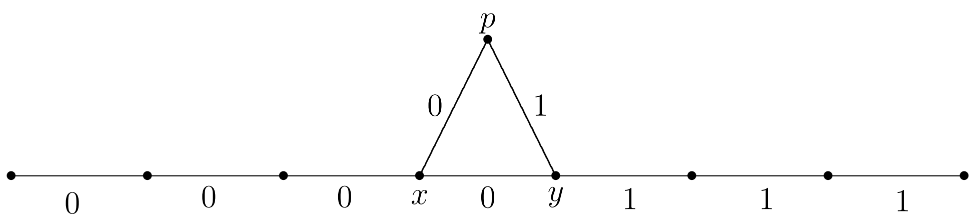
(b)

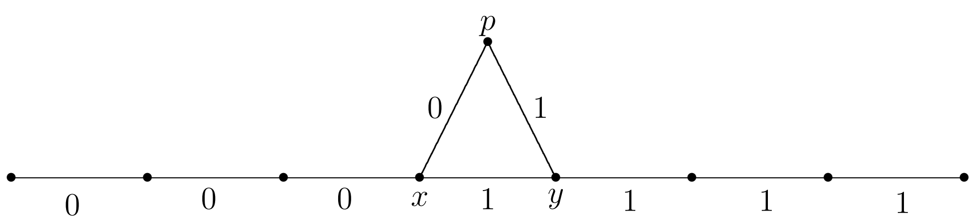


(c) 

In Case (a), we show how to add vertex P to a monotone path of the binary graph on $\{a_1, a_2, \cdots, a_n\} \setminus \{P\}$ to give the given binary graph on $\{a_1, a_2, \cdots, a_n\}$.

If we take out P from (a), we have two cases $1, 2$ for the edge xy since edge xy can be 0 or 1.

(1) 

(2) 

Now both cases 1, 2 show monotone paths for the binary graph on $\{a_1, a_2, \cdots, a_n\} \setminus \{P\}$. In both Case 1 and Case 2, when we add vertex P to the given monotone paths on $\{a_1, a_2, \cdots, a_n\} \setminus \{P\}$ and take out edge xy by the algorithm of Lemma 7, we have the given monotone path specified in Case (a). Case (b) and (c) is similar.

From this, we see that any monotone path for a binary graph on $\{a_1, a_2, \cdots, a_n\}$ can be constructed by the practical construction. ■

7 A Hall Type Theorem

The ultimate goal is to create reasonable necessary and sufficient conditions that determine whether an arbitrary abstract undirected graph on n distinct vertices has a monotone path .

Suppose a_{ij} are assigned to the undirected edges of a complete undirected graph on distinct vertices $\{a_1, a_2, \cdots, a_n\}$.

For each real number x, assign to edge $x_{ij}, i \neq j$, the real number $\overline{a_{ij}} = 1$ if $a_{ij} \geq x$ and assign to x_{ij} the real number $\overline{a}_{ij} = 0$ if $a_{ij} < x$. We call this the derived graph for x. Then the original graph with a_{ij} assigned to the edges has a monotone path if and only if there exists a fixed sequence $x_1, x_2, x_3, \cdots, x_n$ where $\{x_1, x_2, \cdots, x_n\} = \{a_1, a_2, \cdots, a_n\}$ such that this fixed sequence $x_1, x_2, \cdots, x_n$ is a monotone path of the derived graphs for all x.

This extremely primitive theorem at least suggests that a reasonably practical Hall's type theorem may actually exist for the problem under discussion.

Observation 2 We observe that the binary reasoning in this section and in Section 6 was used in Section 3 to easily prove Lemma 3. This reasoning is expanded much further in Section 8.

8 Derived Graphs

Suppose each edge x_{ij} of a complete undirected graph on n distinct vertices $a_1, a_2, \cdots, a_n$ is assigned a real number a_{ij}.

Let $x_1 < x_2 < x_3 < \cdots < x_n$ be arbitrary but fixed real numbers.

We now assign to each edge x_{ij} the real number $\overline{a}_{ij} = 0$ if $a_{ij} < x_1$, assign to x_{ij} the real number $\overline{a}_{ij} = t$ if $x_t \leq a_{ij} < x_{t+1}$ and assign to z_{ij} the real number $\overline{a}_{ij} = n$ if $x_n \leq a_{ij}$.

We call this $\{\overline{a}_{ij}\}$ graph the derived graph for $x_1 < x_2 < \cdots < x_n$. It is easy to see that any monotone path of the original $\{a_{ij}\}$ graph is also a monotone path of the derived $\{\overline{a}_{ij}\}$ graph for any $x_1 < x_2 < \cdots < x_n$. From this it is easy to see that if the original $\{a_{ij}\}$ graph is monotone then the new derived $\{\overline{a}_{ij}\}$ graph is also monotone for all $x_1 < x_2 < \cdots < x_n$.

Therefore, it is necessary that the derived $\{\overline{a_{ij}}\}$ graph be monotone for all $x_1 < x_2 < \cdots < x_n$ in order for the original $\{a_{ij}\}$ graph to be monotone. Also, the original $\{a_{ij}\}$ graph is monotone if and only if for each sequence $x_1 < x_2 < \cdots < x_n$ the new derived $\{\overline{a_{ij}}\}$ graph is monotone.

9 Trinary Graphs

Definition 4 A trinary undirected graph on n vertices is an undirected graph in which each $a_{ij} \in \left\{d, \underline{d}, \overline{d}\right\}$ where $d < \underline{d} < \overline{d}$. We can also let $a_{ij} \in \{0, 1, 2\}$.

Lemma 9 There exists an nonmonotonal trinary graph of $2n+2$ distinct vertices for each $n \geq 1$.

Proof. The proof of Lemma 2 takes care of the case where $n = 1$.

The proof of Lemma 5 can easily take care of the case where $n \geq 2$. Using the drawing and notation given in the proof of Lemma 5, we define $D(x, y)$ as follows.

$D(x, y) = 0$ if $x, y \in \overline{A}$, $D(x, y) = 0$ if $x, y \in \overline{B}$.

$D(x, y) = 1$ if $x \in A, y \in B$.$D\left(\overline{a}, \overline{b}\right) = 1$.

$D(\overline{a}, x) = 2$ if $x \in B$, $D\left(\overline{b}, y\right) = 2$ if $y \in A$. The proof of Lemma 9 and Lemma 5 are now the same. ■

Lemma 10 There exists an nonmonotonal trinary graph of $2n + 3$ distinct vertices for each $n \geq 2$.

Proof. Proof 2 of Lemma 6 can be modified to prove Lemma 10. We let $D(x, y) = 0$ if $x, y \in \overline{A}$, $D(x, y) = 0$ if $x, y \in \overline{B}$.

$D(x, y) = 1$ if $x \in A, y \in \dot{B}$. $D\left(\overline{a}, \overline{b}\right) = 1$, $D\left(a^*, \overline{b}\right) = 1$. $D(\overline{a}, x) = 2$ if $x \in B$, $D(a^*, x) = 2$ if $x \in B$, $D\left(\overline{b}, y\right) = 2$ if $y \in A$.

Proof 2 of Lemma 6 can now be used to prove Lemma 10. ■

Note 4 In section 10 we show that all trinary graphs on $n = 5$ vertices are monotone.

10 A Five-Vertex Graph With Three Distinct Edge Labels Is Monotone

Definition: Given a graph with five vertices A, B, C, D and E with each of its ten edges labeled either $1, 2$ or 3, we say the graph is monotone if the vertices can be permuted into $VWXYZ$ in such a way that the labels on the four successive edges VW, WX, etc. are non-decreasing.

Introductory Observations:

1. The graph is monotone if the vertices can be permuted in such a way that the labels on the four successive edges are non-increasing.

2. If there are three successive edges with the same label, the graph is monotone.

3. If all ten of the edges in the graph have the same label, the graph is monotone. [Obviously something much stronger is true.]

Terminology

1. We call an edge with label 1 a '1-edge'. Similarly, we have 2-edges and 3-edges.

2. We say that a vertex has 1-degree d if there are exactly d 1-edges incident to the vertex. We define 2-degree and 3-degree similarly. Thus, d is an integer between 0 and 4 inclusive, and, for each vertex, the sum of the 1-degree, the 2-degree and the 3-degree must be four.]

Summary

We will show that any graph of the type described above is monotone. Our plan of attack is to show first that if the graph does not have a vertex with 2-degree at least 2, then the graph must be monotone. We accomplish this by considering separately the cases: (1) there is at most one 2-edge in the graph and (2) there are exactly two 2-edges in the graph and these edges are disjoint (so that there is no vertex with 2-degree at least 2).

We then consider graphs which contain at least one vertex with 2-degree at least two and show that all such graphs are monotone. Note here that if the graph contains at least three 2-edges, the graph must contain at least one vertex with 2-degree at least two. Alternatively, the graph might contain exactly two 2-edges which happened to be adjacent.

Lemma 11. If there is at most one 2-edge in the graph, the graph is monotone.

Proof: Since the graph has exactly ten edges and at most one of these is a 2-edge, there must be at least five 1-edges or at least five 3-edges. Without loss of generality, we may assume that there are at least five 1-edges. It follows that the sum of the 1-degrees must be at least ten. Therefore there must be a vertex with 1-degree at least two.

Suppose then that edges AB and BC are both 1-edges. If, with $X = D$ or E, AX or CX were a 1-edge, the graph would be monotone, since we would have three successive 1-edges. So, imagine that none of the edges AD, AE, CD and CE is a 1-edge. Then, since the graph contains at least five 1-edges, at least three of BD, BE, AC and DE must be 1-edges. Thus, at least one of BD and BE is a 1-edge and at least one of AC and DE is a 1-edge.

Suppose that BD is a 1-edge. If AC is a 1-edge, consider the path $ACBD$, which consists of three successive 1-edges so the graph is monotone. Now suppose that BD and DE are 1-edges. Then $ABDE$ consists of three successive 1-edges, and the graph is monotone.

The cases in which BE is a 1-edge are exactly similar. ■

Lemma 12. If there are exactly two 2-edges in the graph and these edges are disjoint, the graph monotone.

Proof: Suppose that AB and CD are the 2-edges. Without loss of generality, we may assume that AC is a 1-edge. Now if both BE and DE are both 3-edges, the path $ACDEB$ solves the graph since the lengths of the successive edges are 1, 2, 3 and 3. Therefore, if the graph is not monotone, either BE or DE must be a 1-edge. Considering the symmetry in the graph, we may suppose, without loss of generality, that BE is a 1-edge.

Now, if EA is a 1-edge, CA, AE and EB are successive 1-edges and the graph is monotone. Similarly, if EC is a 1-edge, AC, CE and EB are successive 1-edges and the graph is monotone. So, we assume that both EA and EC are 3-edges.

Now, if CB is a 1-edge, then EB, BC and CA are successive 1-edges and the graph is monotone. Moreover, if CB is a 3-edge, then BC, CE and EA are successive 3-edges and the graph is monotone.

We may now conclude that if the graph has exactly two 2-edges and these edges are disjoint, the graph is monotone. ■

Lemma 13. A five-vertex graph with edge labels 1, 2 and 3 is monotone.

Proof: Using Lemmas 11 and 12, we see that we need only consider graphs with either at least three 2-edges or exactly two 2-edges which are NOT disjoint. It is easy to see that if there are three or more 2-edges, there must be a vertex with 2-degree at least two. So, we need only consider graphs which contain a vertex with 2-degree at least two. We will assume that vertex B is such a vertex and that AB and BC are 2-edges.

If any one of the edges AX and CX with $X = D$ or E is a 2-edge, the graph would contain three successive 2-edges and would be monotone. So, we may assume that each of these four edges is either a 1-edge or a 3-edge. We assume, without loss of generality, that AE is a 1-edge. If CD is a 3-edge, the graph is monotone with path $EABCD$ so we assume that CD is a 1-edge. Arguing in the same fashion, we see that the edges AD and CE must have the same label. If this common label is 1, then the successive edges AD, DC and CE are all 1-edges and the graph is monotone. So we assume that AD and CE are both 3-edges.

So, to summarize, we are assuming that AB and BC are 2-edges, AE and CD are 1-edges and AD and CE are 3-edges. [Drawing a diagram will help in what follows.] Consider AC. If AC is a 1-edge, EA, AC and CD are successive 1-edges and the graph is monotone. If AC is a 3-edge, DA, AC and CE are successive 3-edges and the graph is monotone. We assume, therefore that AC is a 2-edge. Similarly, if DE is a 1-edge or if DE is a 3-edge, the graph is monotone, so we assume that DE is a 2-edge.

At this point, the only edges whose labels are unknown are BD and BE. Consider BD. If BD is a 2-edge, then AC, CB and BD are successive 2-edges and the graph is monotone. If BD is a 1-edge, then consider $DBACE$, which has successive edge labels 1, 2, 2 and 3 so the graph is monotone. If BD is a 3-edge, then consider $EACBD$, which also has successive edge labels 1, 2, 2 and 3 so the graph is monotone. [We could also have considered the three possible labels for BE and shown that the graph was monotone in each case.]

We conclude that the graph is monotone. ■

11 Graphs with $a_{ij} \in \{0, 1, 2, 3\}$

In this section, we allow the edges x_{ij} of an undirected graph on $n \geq 3$ distinct vertices to be assigned values $a_{ij} \in \{0, 1, 2, 3\}$.

Lemma 14 There exists an nonmonotonal graph on 5 distinct vertices when $a_{ij} \in \{0, 1, 2, 3\}$.

Let a_1, a_2, a_3, a_4, a_5 be the 5 vertices arranged in clockwise order of a regular pentagon. For each a_1, a_2, a_3, a_4, a_5, let us define $a_{ij} = 0$ if $a_i a_j$ is a side of the pentagon and $a_{ij} = 2$ if $a_i a_j$ is a diagonal of the pentagon. A monotone path for a_1, a_2, a_3, a_4, a_5 must consist of consecutive vertices such as a_1, a_2, a_3, a_4, a_5 or alternate vertices such as a_1, a_3, a_5, a_2, a_4.

Let us now modify the above a_{ij}'s as follows to get a new graph. Let $a_{12} = 0, a_{23} = 1, a_{34} = 0, a_{45} = 0, a_{51} = 1, a_{13} = 2, a_{14} = 3, a_{24} = 3, a_{25} = 2, a_{35} = 3$. Now a monotone path , if a monotone path exists, for this new graph must still consist of consecutive vertices or consecutive alternate vertices. It is now easy to see that this new graph is not monotone. ■ The proof above is very similar to the proof of Lemma 3.

12 A Variation

A variation of this problem is to find a maximal length monotone path in an undirected graph. The approach we would take is to modify and generalize rectangularity type diagrams used in the proofs of Lemmas 5 and 6. It seems easy to construct undirected graphs on $n \geq 6$ vertices so that the maximal montone path has at most $\lceil \frac{n+2}{2} \rceil$ vertices. However, we believe that even more can be said.

13 Discussion

For each $m \in \{2, 3, 4, 5, 6, \cdots\}$, it is obvious that Lemmas 9, 10 remain true if we assign to each edge x_{ij} of a complete undirected graph on $n = 4$ or $n \geq 6$ vertices one of the values $a_{ij} \in \{0, 1, 2, \cdots, m\}$. Also, Lemma 14 remains true if we assign to each edge x_{ij} of a complete undirected graph on $n = 5$ vertices one of the values $a_{ij} \in \{0, 1, 2, 3, \cdots, m\}$ where $m \in \{3, 4, 5, 6, \cdots\}$.

Acknowledgements. The monotonicity definition in this paper originated with Patrick Vennebush, with whom we had several informative discussions.

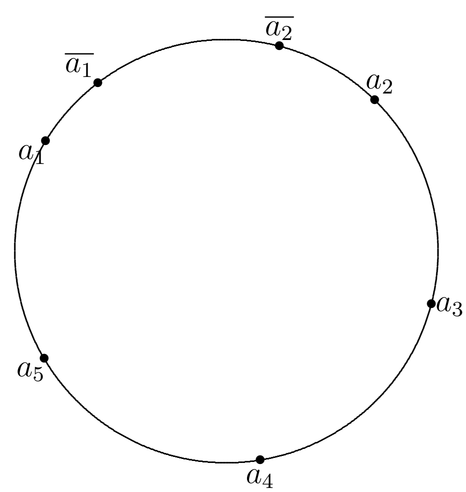
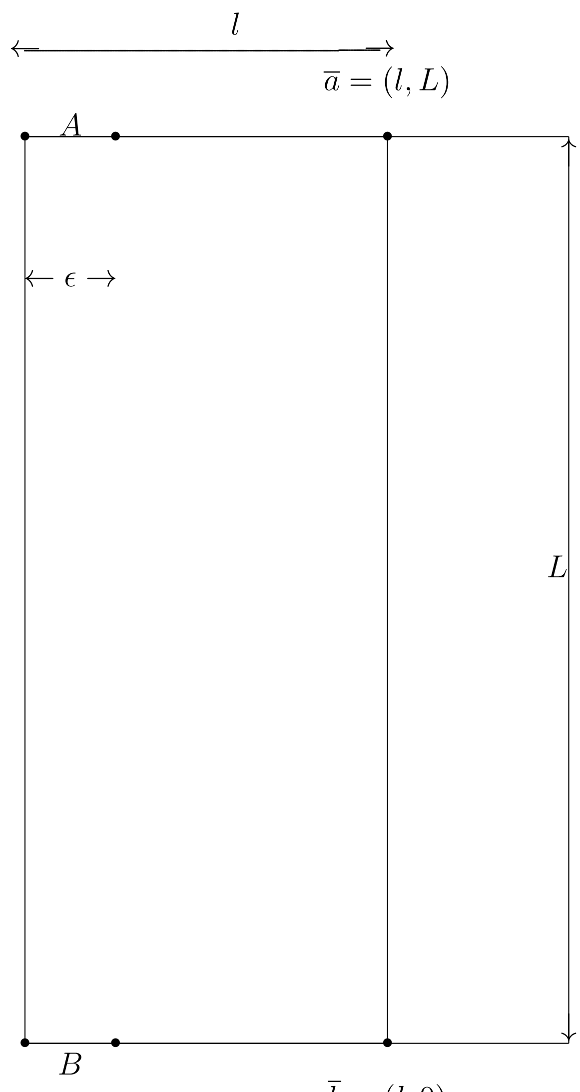


Fig. 1


l
$\overline{a} = (l, L)$
A
$\leftarrow \epsilon \rightarrow$
L
B
$\overline{b} = (l, 0)$
Fig. 2

References

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