Groups that Distribute over Stars

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1 Abstract

Suppose $(S,*)$ is a mathematical structure on a set S. As examples, $(S,*)$ might be a topological space on S, a topological group on S, an n-ary operator on S, an n-ary relation on S or a Steiner triple system on S. A similarity mapping on $(S,*)$ is a permutation on S that preserves the structure of $(S,*)$. Such mappings $f:(S,*)\rightarrow(S,*)$ are called by different names. As examples, f might be called a homeomorphism or an automorphism on $(S,*)$. See [5] for the details.

Suppose $(S,\cdot)$ is a group on S. For each fixed t in S, the left and right translation on S are the permutations $L_t(x)=t\cdot x$ and $R_t(x)=x\cdot t$ respectively. They are called the left and right translation by t. See [1] and [5].

We say that the group $(S,\cdot)$ left-distributes or right-distributes over $(S,*)$ if respectively for all t in $S, L_t(x):(S,*)\rightarrow(S,*)$ or $R_t(x):(S,*)\rightarrow(S,*)$ is a similarity mapping on $(S,*)$. In case $S=(\mathbb{R},0,+)$, the additive group of real numbers with the usual topology, addition both left and right distributes over the topological space $(\mathbb{R},T)$ because, for every open set U and every $x\in\mathbb{R}$, $U+x=x+U$ is an open set. See [5] for more examples.

In this paper, we give a naturally occurring example that involves what we call an n-star which is structurally the same as n lines in the plane intersecting in $\binom{n}{2}$ distinct points. However, an equally important purpose of this paper is to show that if a structure $(S,*)$ is given, then a fundamental idea is to see if a group $(S,\cdot)$ exists such that $(S,\cdot)$ left-distributes or right-distributes over $(S,*)$. The only prerequisites for reading this paper is a very basic knowledge of set theory, group theory and the mod p field $\mathbf{Z}_p=(\{0,1,2\ldots,p-1\},0,1,+,\cdot)$, where p is a prime.

2 Introduction

For an arbitrary set S, when you think about a structure $(S,*)$ on S, you usually think about things like binary operators, binary relations, topological spaces, Boolean algebras, Steiner triples, etc.

But if S is given, then a structure $(S,*)$ on S can actually be anything you can imagine that uses the elements of S. More specifically, $(S,*)$ can be any set that one can define that uses the elements of S, and that is pretty big. As an example, if $S=\{a,b,c\}$, then $(S,*)$ might be something like $(S,*)=(\{a,b,c\},\{\{a,b\},b,c,\{a,c\}\})$ when $*=\{\{a,b\},b,c,\{a,c\}\}$. In this example, we call a,b, and c the atoms of S. A similarity mapping on $(S,*)$ is a permutation for S that preserves the structure of $(S,*)$. Using the above $(S,*)$ as an example, basically what this means is that if the atoms a,b,c in $*$ are replaced by $f(a),f(b),f(c)$ respectively then $*$ remains unchanged. Thus, the permutation f on S defined by $f(a)=a, f(b)=c, f(c)=b$ is a similarity mapping on $(S,*)$.

Many readers know that a similarity mapping f on a topological space $(S,*)=(S,T)$ is called a homeomorphism. This means that for all $A\subseteq S, A\in T$ is true if and only if $f(A)\in T$. Also, a similarity mapping f for a binary operator $(S,*)$ is called an automorphism on $(S,*)$. This means that for all $a,b\in S, f(a*b)=f(a)*f(b)$. Also, a similarity mapping f for a binary relation $(S,*)=(S,R)$ is called simply a similarity mapping, and this means that for all $a,b\in s, aRb$ is true if and only if $f(a)Rf(b)$.

The collection of all similarity mappings for a structure $(S,*)$ forms a group under composition of functions. This is because if f,g are similarity mappings on $(S,*)$, then $f\circ g, g\circ f, f^{-1}$ and g^{-1} are also similarity mappings on $(S,*)$. Also, the identity permutation I on S, is always a similarity mapping on $(S,*)$. In the next section, we give a motivating example behind the theory in this paper. Then we define what it means to say that a group $(S,\cdot)$ left-distributes or right-distributes over any given structure $(S,*)$ on this set S. Then for any structure $(S,*)$ on a set S, we can always ask the fundamental questions "Does there exist a group $(S,\cdot)$ on S such that $(S,\cdot)$ left-distributes or right-distributes over $(S,*)$ (or both)? Also, if such groups $(S,\cdot)$ exist on S, find them all". In the remainder of the paper we deal with a specific example called an n-star.

3 A Motivating Example

Let us now observe the following drawing, familiar to those readers who have studied projective geometry. You might also want to look up the definitions of a "block design" and a "Steiner triple system".

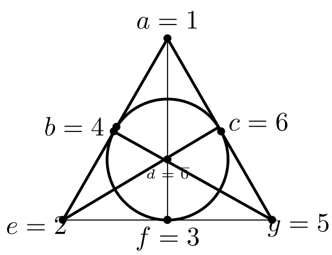


Fig. 1. A seven line, seven point figure.

As in projective geometry we define the following seven lines: $l_1=\{1,2,4\}, l_2=\{0,1,3\}, l_3=\{1,5,6\}, l_4=\{0,4,5\}, l_5=\{0,2,6\}, l_6=\{2,3,5\}, l_7=\{3,4,6\}$. Of course, line l_7 is the three points on the circle. We observe that any two distinct points $p,\overline{p}$ lie on a unique line and any two distinct lines $l,\overline{l}$ intersect in a unique point. Let us define the usual mod 7 cyclic group $(\{0,1,2,3,4,5,6\},0,+)$ where addition is carried out mod 7. Thus, $2+4=6, 4+6=3$, etc. Thus, we have defined both a group $(S,0,+)=(\{0,1,2,3,4,5,6\},0,+)$ and a structure $(S,*)=(\{0,1,2,3,4,5,6\},\{l_1,l_2,l_3,l_4,l_5,l_6,l_7\})$ on the same set S.

By experimenting you can convince yourself that for all fixed $t\in S$, the permutation $P_t=\{(x,t+x): x\in S\}$ is a similarity mapping on $(S,*)$. This is because for all $t\in S, t+\{l_1,l_2,\cdots,l_7\}=\{t+l_1,t+l_2,\cdots,t+l_7\}=\{l_1,l_2,\cdots,l_7\}$ which means that P_t maps lines to lines. For example, $3+l_1=\{3+1,3+2,3+4\}=\{4,5,0\}=l_4, 3+l_2=\{3,4,6\}=l_7, 5+l_3=\{4,1,2\}=l_1,\cdots,3+l_7=\{6,0,2\}=l_5$.

We say that this group $(S,0,+)$ distributes over this structure $(S,*)$. We generalized this example in a completely straightforward way so that the above idea applies to any structure $(S,*)$ on S. So you now know most of what we knew when we started our work on groups that distribute over mathematical structures. We have since found other examples such as Howard Eves' abstract equihoop, idempotent Abelian (or medial) quasigroups, (other) Steiner triples and topological spaces that have groups that left (or right) distribute over them.

In this paper, we give yet another example called n-stars. Remember, a structure $(S,*)$ is anything that we can imagine on a set S. We hope that by adding these basic ideas to your collection of ideas then perhaps you may be able to add to this collection as well. The success of this theory depends on finding many examples.

In the rest of this paper, we use the notation $\exists$ to mean "there exists" and $\forall$ to mean "for every". Thus, the sentence "$\forall\varepsilon > 0 \exists\sigma > 0$ such that $|f(x+\sigma) - f(x)| < \varepsilon$" means, For every ε greater than zero, there exists a σ greater than zero such that $|f(x+\sigma) - f(x)| < \varepsilon$.

4 Groups that Distribute over Mathematical Structures

Suppose $(S, \cdot)$ is a group on a set S. For each fixed t in S, the left and right translations on S are the permutations $L_t(x) = t \cdot x$ and $R_t(x) = x \cdot t$ respectively. They care called the left and right translations by t. See [1] and [5].

Motivated by the previous example, we say that the group $(S, \cdot)$ left-distributes or right-distributes over any arbitrary mathematical structure $(S, *)$ on the same set S if respectively for all t in $S, L_t(x) : (S, *) \to (S, *)$ or $R_t(x) : (S, *) \to (S, *)$ is a similarity mapping on $(S, *)$. Also, $(S, \cdot)$ distributes over $(S, *)$ if it both left and right distributes over $(S, *)$. As a specific example, suppose $(S, *)$ is a binary operator on S. Then $(S, \cdot)$ left-distributes over $(S, *)$ if $\forall t \in S, L_t(x) = t \cdot x$ is an automorphism on $(S, *)$. This means that $\forall t \in S, \forall a, b \in S, L_t(a * b) = (L_t(a)) * (L_t(b))$ which means that $\forall t \in S, \forall a, b \in S, t \cdot (a * b) = (t \cdot a) * (t \cdot b)$.

Likewise, $(S, \cdot)$ left-distributes over the binary relation $(S, *) = (S, R)$ if $\forall t \in S$ and $\forall a, b \in S$, aRb is true if and only if $(t \cdot a) R (t \cdot b)$.

5 Background Material

Suppose $(S, *)$ is any structure on S. In the paper [5] which the reader can easily access electronically, we show how to construct all groups $(S, \cdot)$ on S such that $(S, \cdot)$ left-distributes (or right-distributes) over $(S, *)$ if such a group $(S, \cdot)$ exists. However, usually such a group $(S, \cdot)$ will not exist. Our construction used the group of all similarity mappings on $(S, *)$ which we called $(F, \circ)$. We showed that a group $(S, \cdot)$ exists such that $(S, \cdot)$ left-distributes over $(S, *)$ if and only if there exists a subgroup $\left(\overline{F}, \circ\right)$ of $(F, \circ)$ such that $\left(\overline{F}, \circ\right)$ is uniquely transitive on S. This means that for every $a, b \in S$, there exists a unique $f \in \overline{F}$ such that $f(a) = b$. If such a $\left(\overline{F}, \circ\right)$ exists, then a group $(S, \cdot)$ that left-distributes over $(S, *)$ was defined in theorem 5, [5] as follows, and the reader can prove this for himself.

First, we arbitrarily choose $1 \in S$ to be the identity of $(S, \cdot)$. Then we index $\overline{F} = \{f_t : t \in S\}$ such that $\forall i \in S, f_i(1) = i$. We can do this since $\left(\overline{F}, \circ\right)$ is uniquely transitive on S. A group $(S, \cdot)$ with identity 1 that left-distributes over $(S, *)$ is defined by $\forall i, j \in S, i \cdot j = f_i(j)$. It is also obvious that if the group $(S, \cdot)$ left-distributes over $(S, *)$ then the group $(S, \odot)$ defined by $a \odot b = b \cdot a$ will right-distribute over $(S, *)$.

From the transitive property of $(\overline{F}, \circ)$, it is obvious that intuitively a necessary condition on $(S, *)$ is that the structure of $(S, *)$ must be very homogeneous and symmetric. We cannot tell by just looking at a structure $(S, *)$ whether it is homogenous and symmetric enough or not. However, any time we encounter a structure $(S, *)$ that appears to be fairly homogeneous and symmetric, it is natural to ask if a group $(S, \cdot)$ exists which left (or right) distributes over $(S, *)$.

We now proceed to illustrate this by studying n-stars. Intuitively these stars are homogeneous and symmetric and they also look alike. But some have groups that left (or right) distributes over them and some do not, which illustrates the delicate balance that must exist.

6 Generalized n-stars

Suppose n lines in the plane called $\{l_0, l_1, \cdots, l_{n-1}\} = \{0, 1, 2, \cdots, n-1\} = [0, n-1]$ intersect each other in $\binom{n}{2} = \frac{n(n-1)}{2}$ distinct points which we variously call $\{x_1, x_2, \cdots, x_{\frac{n(n-1)}{2}}\} =$

$\{\{l_i,l_j\}: i\neq j, i,j\in\{0,1,2,\cdots,n-1\}\} = \{\{i,j\}: i\neq j, i,j\in\{0,1,2,\cdots,n-1\}\} = D_{[0,n-1]}$. D stands for the doubleton sets on $[0,n-1]$.

If these n lines are the sides of a regular n-gon, then these $\binom{n}{2}$ points can be viewed as generalized n-stars; however, we must allow points at infinity when n is even. In Fig. 2 we show the n-stars for $n=3,4,5,6$ and in Fig. 3 we show the n-star for $n=7$. Denote the n-star by $\left(D_{[0,n-1]},*\right)$.

Let us now define the group of all permutations on $\{l_0,l_1,\cdots,l_{n-1}\}=\{0,1,2,\cdots,n-1\}=[0,n-1]$ using composition of functions. This group which contains $n!$ permutations is the standard symmetric group on $[0,n-1]$.

Each permutation f on $[0,n-1]$ defines a corresponding line-preserving permutation $\overline{f}$ on $D_{[0,n-1]}$, $\overline{f}(\{i,j\}) = \{f(i),f(j)\}$. For example, if $f=\begin{pmatrix} l_0 & l_1 & l_2 & l_3 & l_4 \\ l_2 & l_0 & l_4 & l_1 & l_3\end{pmatrix}=\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 0 & 4 & 1 & 3\end{pmatrix}$, then

$\overline{f}=\begin{pmatrix} \{0,1\} & \{0,2\} & \{0,3\} & \{0,4\} & \{1,2\} & \{1,3\} & \{1,4\} & \{2,3\} & \{2,4\} & \{3,4\} \\ \{0,2\} & \{2,4\} & \{1,2\} & \{2,3\} & \{0,4\} & \{0,1\} & \{0,3\} & \{1,4\} & \{3,4\} & \{1,3\}\end{pmatrix}$. The above line preserving permutation $\overline{f}$ is shown in Fig. 4. The reader should study this carefully.

In Fig. 4 we note that line l_0 is moved to l_2, line l_1 is moved to l_0, line l_2 is moved to l_4, line l_3 is moved to l_1 and line l_4 is moved to l_3. This changes the positions of the points $\{i,j\}$ as shown. The important thing to notice is that if 4 points in the first drawing lie in a straight line, then these same 4 points lie in a straight line in the second drawing. For example, $\{0,1\},\{0,2\},\{0,3\},\{0,4\}$ lie on line l_0 in the first drawing and these same 4 points lie on line l_0 in the second drawing. This is why we call $\overline{f}$ a line preserving permutation on $D_{[0,n-1]}$. This is similar to the line preserving ideas discussed in Fig. 1.

In Lemmas 1, 2, we show that these $n!$ line preserving permutations on $D_{[0,n-1]}$ form a group using composition of functions. The following Lemmas 1, 2 also relate this group of line preserving permutations on $D_{[0,n-1]}$ and the symmetric group on $[0,n-1]$.

Lemma 1. Suppose f,g are permutations on $[0,n-1]$, $n\geq 3$, and $\overline{f},\overline{g}$ are the corresponding line preserving permutations on $D_{[0,n-1]}$. Then $f\neq g$ implies $\overline{f}\neq\overline{g}$. Thus, the mapping $f\to\overline{f}$ is one to one.

Proof. Since $f\neq g, \exists\ a\in[0,n-1]$ such that $f(a)\neq g(a)$.

Suppose $b\in[0,n-1]\setminus\{a\}$. Now if $\overline{f}(\{a,b\})=\{f(a),f(b)\}\neq\overline{g}(\{a,b\})=\{g(a),g(b)\}$ then there is nothing to prove. Therefore, suppose $\{f(a),f(b)\}=\{g(a),g(b)\}$. Thus $f(a)\neq g(a)$ implies $f(a)=g(b)$ and $f(b)=g(a)$.

Suppose $c\in[0,n-1]\setminus\{a,b\}$. Now, $g(c)\notin\{g(a),g(b)\}=\{f(a),f(b)\}$. Also, $\overline{f}(\{a,c\}=\{f(a),f(c)\}$ and $\overline{g}(\{a,c\}=\{g(a),g(c)\}$. Since $f(a)\neq g(a)$ and $f(a)\neq g(c)$ we see that $\{f(a),f(c)\}\neq\{g(a),g(c)\}$ which means $\overline{f}(\{a,c\})\neq\overline{g}(\{a,c\})$. Therefore, $\overline{f}\neq\overline{g}$. ■

Lemma 2. If f,g are permutations on $[0,n-1]$ and $\overline{f},\overline{g}$ are the corresponding line preserving permutations on $D_{[0,n-1]}$ then $\overline{f\circ g}=\overline{f}\circ\overline{g}$.

Note 1. Thus, the symmetric group of all permutations on $[0,n-1]$, $n\geq 3$, is isomorphic to the corresponding group of line preserving permutations on $D_{[0,n-1]}$.

Proof of Lemma 2. $\left(\overline{f\circ g}\right)(\{i,j\})=\{(f\circ g)(i),(f\circ g)(j)\}$. Also, $\left(\overline{f}\circ\overline{g}\right)(\{i,j\})=\overline{f}(\overline{g}(\{i,j\}))=\overline{f}(\{g(i),g(j)\})=\{f(g(i)),f(g(j))\}=\{(f\circ g)(i),(f\circ g)(j)\}$. Therefore, $\overline{f\circ g}=\overline{f}\circ\overline{g}$. ■

7 Groups that Distribute over n-stars

As in the Fig. 1 drawing, we say that a permutation $\overline{f}$ on $D_{[0,n-1]}$ is a similarity mapping on the n-star $\left(D_{[0,n-1]},*\right)$ if and only if $\overline{f}$ maps lines onto lines. It is easy to show that this is true if and only if $\overline{f}$ corresponds to some permutation f on $\{l_0,l_1,\cdots,l_{n-1}\}=[0,n-1]$ as defined above. The reason for this is that the $n-1$ points on the line l_0 can be mapped in $(n-1)!$ different ways onto the $n-1$ points of any line l_i. Once the $n-1$ points on line l_0 have been mapped onto l_i, the mapping of the other points of the n-star are uniquely determined from this. This gives a total of $n\cdot(n-1)!=n!$ different mappings which is the same number as the $n!$ permutations on $\{l_0,l_1,\cdots,l_{n-1}\}$.

As noted in section 4, this means that a group $\left(D_{[0,n-1]},\cdot\right)$ with operator $(\cdot)$ on the set $D_{[0,n-1]}$ left-distributes over the n-star $\left(D_{[0,n-1]},*\right)$ if and only if for all fixed $t\in D_{[0,n-1]}$ the permutation $\left\{(x_i,t\cdot x_i):x_i\in D_{[0,n-1]},\right\}$ is a line preserving permutation on $D_{[0,n-1]}$.

If we examine the 5 stars shown in Fig. 2 and Fig. 3, we see that the points on these 5 stars intuitively seem to be fairly homogeneous and symmetric. As always in such a case, it is natural to ask if there exists a group that left (or right) distributes over the n-stars $\left(D_{[0,n-1]},*\right)$ as n ranges over $\{3,4,5,6,\cdots\}$.

The following Lemmas 3, 4 partially answer this question.

Lemma 3. If $\frac{n(n-1)}{2}$ if even, then there does not exist a group $\left(D_{[0,n-1]},\cdot\right)$ that left (or right) distributes over the n-star $\left(D_{[0,n-1]},*\right)$.

Proof. Let $(F,\circ)$ be the group of all line preserving permutations on $D_{[0,n-1]}$. Also, suppose there exists a subgroup $\left(\overline{F},\circ\right)$ of $(F,\circ)$ such that $\left(\overline{F},\circ\right)$ is uniquely transitive on $D_{[0,n-1]}$. Of course, $\overline{F}$ must have exactly $\left|\overline{F}\right|=\frac{n(n-1)}{2}$ permutations since $\left(\overline{F},\circ\right)$ is uniquely transitive on $D_{[0,n-1]}$. Since $\left|\overline{F}\right|$ is even, by the Syloe theorems of group theory, we know that $\exists\overline{f}\in\overline{F}$ such that $\overline{f}\neq i$ and $\overline{f}\circ\overline{f}=i$, the identity permutation on $D_{[0,n-1]}$. By the isomorphism $\overline{f\circ g}=\overline{f}\circ\overline{g}$ stated in Lemma 2, this means that $\exists$ a permutation f on $[0,n-1]$ such that (1) $f\neq I$, (2) $f\circ f=I$, the identity permutation on $[0,n-1]$, and (3) $f,\overline{f}$ correspond to each other as defined earlier by $\overline{f}\left(\{i,j\}\right)=\{f(i),f(j)\}$.

Since $f\neq I$ and $f\circ f=I$, this implies that $\exists i,j\in[0,n-1],i\neq j$, such that $f(i)=j$ and $f(j)=i$.

Therefore, $\overline{f}\left(\{i,j\}\right)=\{f(i),f(j)\}=\{i,j\}$. But since $\overline{f}\neq i$, the identify permutation on $D_{[0,n-1]}$ and since $i\left(\{i,j\}\right)=\{i,j\}$ this means that $\left(\overline{F},\circ\right)$ cannot be uniquely transitive on $D_{[0,n-1]}$ since $\overline{f}\left(\{i,j\}\right)=i\left(\{i,j\}\right)=\{i,j\}$. ■

Corollary 1. Suppose p is a prime of the form $p=4k+1$. Then there does not exist a group $\left(D_{[0,n-1]},\cdot\right)$ that left (or right) distributes over the p-star $\left(D_{[0,p-1]},*\right)$.

Proof. Note that $\frac{p(p-1)}{2}=2k\left(4k+1\right)$ is even, and Lemma 3 completes the proof. ■

Lemma 4. Suppose p is a prime of the form $p=4k+3$. Then $\exists$ a group $\left(D_{[0,p-1]},\cdot\right)$ that left distributes over the p-star $\left(D_{[0,p-1]},*\right)$. We now develop the machinery to prove Lemma 4. This machinery is just a generalization of the work on the 7-star.

8 Some Basic Number Theory

If p is a prime, let $\mathbf{Z}_p=\left(\{0,1,2,\cdots,p-1\},0,1,+,-,\cdot,\div\right)=\left([0,p-1],0,1,+,-,\cdot,\div\right)$ denote the standard mod p field on $[0,p-1]$. The following theorem can be found in any elementary number theory book.

Theorem 1. Suppose p is a prime. Then in the field $\mathbf{Z}_p$, the subgroup $\left(\{1,2,\cdots,p-1\},\cdot\right)=\left([1,p-1],\cdot\right)$ is a cyclic group with identity 1. This means that $\exists m\in[1,p-1]$ such that $\left\{m,m^2,m^3,\cdots,m^{p-1}=1\right\}=[1,p-1]$.

Corollary 2. Suppose p is an odd prime. Since $2\mid p-1$ and since $\left([1,p-1],\cdot\right)$ is a cyclic group, $\exists m\in[1,p-1]$ such that $m^{\frac{p-1}{2}}=1$ and the elements of the set $\left\{m,m^2,m^3,\cdots,m^{\frac{p-1}{2}}=1\right\}$ are all distinct.

Lemma 5. Suppose p is a prime of the form $p=4k+3$. Therefore, $\frac{p-1}{2}$ is odd. Also, suppose $m\in[1,p-1]$ satisfies $m^{\frac{p-1}{2}}=1$ and the elements of the set $\left\{m,m^2,m^3,\cdots,m^{\frac{p-1}{2}}=1\right\}$ are all distinct. Then, $\forall t\in[1,p-1],m^t\neq-1$ in $\mathbf{Z}_p$. That is, $m^t\not\equiv-1(\text{mod }p)$.

Proof. Since $m^{\frac{p-1}{2}}=1$, we only have to show that $\forall t\in\left[1,\frac{p-1}{2}-1\right],m^t\neq-1$ in $\mathbf{Z}_p$. Therefore, assume $t\in\left[1,\frac{p-1}{2}-1\right]$ and $m^t=-1$. This implies $m^{2t}=1$. Now $2t\neq\frac{p-1}{2}$ since $\frac{p-1}{2}$ is odd. Now if $2t<\frac{p-1}{2}$, then $m^{2t}=m^{\frac{p-1}{2}}=1$ contradicts the hypothesis that $\left\{m,m^2,\cdots,m^{\frac{p-1}{2}}=1\right\}$ are all distinct. Suppose $\frac{p-1}{2}<2t<p-1$. Since $m=m^{\frac{p-1}{2}+1},m^2=m^{\frac{p-1}{2}+2},m^3=m^{\frac{p-1}{2}+3},\cdots,m^{\frac{p-1}{2}}=m^{\frac{p-1}{2}+\frac{p-1}{2}}=1,m^{2t}=1$ and $\frac{p-1}{2}<2t<\frac{p-1}{2}+\frac{p-1}{2}$, we again have a contradiction to the hypothesis that $\left\{m,m^2,\cdots,m^{\frac{p-1}{2}}=1\right\}$ are all distinct. ■

Suppose p is an odd prime. $\forall x,y\in[0,p-1]$, we define the diameter of the set $\{x,y\}$ as $D\left(\{x,y\}\right)=\{x-y,y-x\}$ where the calculations are in $\mathbf{Z}_p$.

Lemma 6. Suppose p is an odd prime. Then the following is true in $\mathbf{Z}_p$.

(a) $\forall x, y \in [0, p-1]$ if $x = y$ then $D(\{x, y\}) = \{0\}$. If $x \neq y$ then $D(\{x, y\}) = \{x - y, y - x\}$ is a doubleton subset of $[1, p-1]$.

(b) $\forall x, y \in [0, p-1]$, if $x \neq y$ then one member of $D(\{x, y\})$ lies in $[1, \frac{p-1}{2}]$ and the other member lies in $\left[\frac{p+1}{2}, p-1\right]$. Item (b) is not used in this paper.

(c) $\forall x, y, \overline{x}, \overline{y} \in [0, p-1]$, $D(\{x, y\}) = D(\{\overline{x}, \overline{y}\})$ or $\mathrm{D}(\{x, y\}) \cap D(\{\overline{x}, \overline{y}\}) = \phi$.

(d) Suppose $x, y, \overline{x}, \overline{y} \in [0, p-1]$ and $D(\{x, y\}) = D(\{\overline{x}, \overline{y}\})$. Then $\exists$ a unique $t \in [0, p-1]$ such that $\{x, y\} + t = \{x + t, y + t\} = \{\overline{x}, \overline{y}\}$.

Proof. (a) Suppose $x \neq y$ and $x - y = y - x$. This implies $2x = 2y$ in $\mathbf{Z}_p$ which is impossible since p is an odd prime.

(b) Now $x - y, y - x \in [1, p-1]$ since $x \neq y$. Also, $(x - y) + (y - x) = 0$ in $\mathbf{Z}_p$. Therefore, $(x - y) \dot{+} (y - x) = p$ where $\dot{+}$ is the usual addition. The conclusion follows from this.

(c) Now $D(\{x, y\}) = \{x - y, y - x\}$ and $D(\{\overline{x}, \overline{y}\}) = \{\overline{x} - \overline{y}, \overline{y} - \overline{x}\}$. Suppose $D(\{x, y\}) \cap D(\{\overline{x}, \overline{y}\}) \neq \phi$. If $x - y = \overline{x} - \overline{y}$ then $y - x = \overline{y} - \overline{x}$ and if $x - y = \overline{y} - \overline{x}$ then $y - x = \overline{x} - \overline{y}$. Also, if $y - x = \overline{x} - \overline{y}$, then $x - y = \overline{y} - \overline{x}$ and if $y - x = \overline{y} - \overline{x}$, then $x - y = \overline{x} - \overline{y}$.

(d) We first prove that there exists at least one $t \in [0, p-1]$ such that $\{x + t, y + t\} = \{\overline{x}, \overline{y}\}$. Since $\{x - y, y - x\} = \{\overline{x} - \overline{y}, \overline{y} - \overline{x}\}$ by symmetry let us suppose $x - y = \overline{x} - \overline{y}$. Therefore, $x - \overline{x} = y - \overline{y} = -t$. Therefore, $x + t = \overline{x}$ and $y + t = \overline{y}$. Next, suppose
$\exists t, \overline{t} \in [0, p-1], t \neq \overline{t}$, such that $\{x + t, y + t\} = \{x + \overline{t}, y + \overline{t}\}$. Therefore, $\{x + t - \overline{t}, y + t - \overline{t}\} = \{x, y\}$. Since $t - \overline{t} \neq 0$ we must have $x + t - \overline{t} = y$ and $y + t - \overline{t} = x$. Therefore, $x - y = \overline{t} - t = t - \overline{t}$. Therefore, $2\overline{t} = 2t$ in $\mathbf{Z}_p$ which is impossible since p is an odd prime, and $t \neq \overline{t}$. ■

Lemma 7. Suppose p is a prime of the form $p = 4k+3$. By corollary 2, let $m \in [1, p-1]$ satisfy $m^{\frac{p-1}{2}} = 1$ and the elements of the set $\left\{m, m^2, m^3, \cdots, m^{\frac{p-1}{2}} = 1\right\}$ are all distinct. Suppose $i \neq j$ and $i, j \in [0, p-1]$ are arbitrary but fixed. Then $D(\{m^y \cdot i, m^y \cdot j\}), y = 0, 1, 2, \cdots, \frac{p-1}{2} - 1$, (where all operations are in $\mathbf{Z}_p$) are pairwise disjoint doubleton subsets of $[1, p-1]$. This implies that these $\frac{p-1}{2}$ doubleton sets partition $[1, p-1]$.

Proof. First, we show that each $D(\{m^y \cdot i, m^y \cdot j\})$ is a doubleton subset of $[1, p-1]$. We show that for all y in $[0, \frac{p-1}{2} - 1]$, $m^y \cdot i \neq m^y \cdot j$, which by Lemma 6a implies that $D(\{m^y \cdot i, m^y \cdot j\})$ is a doubleton subset of $[1, p-1]$. Suppose $m^y \cdot i = m^y \cdot j$. This implies $m^y \cdot (i - j) = 0$ in $\mathbf{Z}_p$ which implies $i - j = 0$ since $m^y \in [1, p-1]$. Therefore, $i = j$ which is a contradiction. Suppose y and $\overline{y}$ are different members of $[0, \frac{p-1}{2} - 1]$. We show that $D(\{m^y \cdot i, m^y \cdot j\}) \cap D\left(\left\{m^{\overline{y}} \cdot i, m^{\overline{y}} \cdot j\right\}\right) = \phi$. Now $D(\{m^y \cdot i, m^y \cdot j\}) = \{m^y \cdot (i - j), m^y \cdot (j - i)\}$.

Also, $D\left(\left\{m^{\overline{y}} \cdot i, m^{\overline{y}} \cdot j\right\}\right) = \left\{m^{\overline{y}} \cdot (i - j), m^{\overline{y}} \cdot (j - i)\right\}$.

First, we show that $m^y \cdot (i - j) \neq m^{\overline{y}} \cdot (i - j)$ in $\mathbf{Z}_p$. Now if $m^y \cdot (i - j) = m^{\overline{y}} \cdot (i - j)$ then since $i \neq j$ we have $m^y = m^{\overline{y}}$. But this is impossible since $y \neq \overline{y}$ and the elements of the set $\left\{m, m^2, \cdots, m^{\frac{p-1}{2}} = 1\right\}$ are all distinct and $m^0 = 1$.

Second, we show that $m^y \cdot (i - j) \neq m^{\overline{y}} \cdot (j - i)$. Now if $m^y \cdot (i - j) = m^{\overline{y}} \cdot (j - i)$ then $\left(m^y + m^{\overline{y}}\right)(i - j) = 0$ in $\mathbf{Z}_p$. But since $i \neq j$ this implies $m^y + m^{\overline{y}} = 0$ in $\mathbf{Z}_p$. Since $y \neq \overline{y}$, by symmetry suppose that $y < \overline{y}$. Therefore, $m^y \left(1 + m^{\overline{y} - y}\right) = 0$ in $\mathbf{Z}_p$ which implies that $m^{\overline{y} - y} = -1$, in $\mathbf{Z}_p$. But since $\overline{y} - y \in \left[1, \frac{p-1}{2} - 1\right]$, by Lemma 5 this is impossible. ■

9 Some Algebraic Machinery

Throughout the rest of this paper, p is a prime of the form $4k + 3$. Also, $m \in [1, p-1]$ satisfies $m^{\frac{p-1}{2}} = 1$ and the elements of the set $\left\{m, m^2, \cdots, m^{\frac{p-1}{2}} = 1\right\}$ are all distinct.

For each fixed $x \in [0, p-1]$, and each fixed $y \in \left[0, \frac{p-1}{2} - 1\right]$ we define the permutation $f_{(x,y)}$ on $[0, p-1]$ by $\forall t \in [0, p-1]$, $f_{(x,y)}(t) = m^y \cdot t + x$ in $\mathbf{Z}_p$. Since $m^{\frac{p-1}{2}} = 1$ we also agree that $\forall y, \overline{y} \in \left[0, \frac{p-1}{2} - 1\right]$, $m^y \cdot m^{\overline{y}} = m^{y + \overline{y}}$ in $\mathbf{Z}_p$ where $y + \overline{y} = y^*$ is defined so that $y^* \in \left[0, \frac{p-1}{2} - 1\right]$ and $y + \overline{y} \equiv y^*, \bmod \frac{p-1}{2}$. Also,

$m^0 = 1$. We need to point out that $y + \overline{y}$ (and later $-y$) are the only operators we will use that are not carried out in the field $\mathbf{Z}_p$. We also do not forget that $\{l_0, l_1, \cdots, l_{p-1}\} = [0, p-1]$ and
$\{\{l_i, l_j\} : i \neq j, i, j \in \{0, 1, 2, \cdots, p-1\}\} = \{\{i, j\} : i \neq j, i, j \in [0, p-1]\} = D_{[0,p-1]}$.

The above statements form the common hypothesis of Lemmas 8-12.

Lemma 8. All of the $\frac{p(p-1)}{2}$ permutations $f_{(x,y)}(t) = m^y \cdot t + x$ on $[0, p-1]$, where $x \in [0, p-1]$, $y \in \left[0, \frac{p-1}{2} - 1\right]$, are distinct.

Proof. Suppose $(x, y) \neq (\overline{x}, \overline{y})$. Now if $x \neq \overline{x}$ then $f_{(x,y)}(0) \neq f_{(\overline{x},\overline{y})}(0)$ which implies $f_{(x,y)} \neq f_{(\overline{x},\overline{y})}$. Therefore, suppose $x = \overline{x}, y \neq \overline{y}$. Now if $f_{(x,y)}(t) = f_{(\overline{x},\overline{y})}(t)$ we must have $\forall t \in [0, p-1], \left(m^y - m^{\overline{y}}\right) \cdot t = 0$ in $\mathbf{Z}_p$. Therefore, $m^y = m^{\overline{y}}$ which is impossible, since the elements of the set $\left\{m^0 = 1, m, m^2, \cdots, m^{\frac{p-1}{2}-1}\right\}$ are all distinct. ■

Lemma 9. $f_{(x,y)} \circ f_{(\overline{x},\overline{y})} = f_{(m^y \cdot \overline{x} + x, y + \overline{y})}$ where $\circ$ is the composition of function,
$m^y \cdot \overline{x} + x$ is carried out in $\mathbf{Z}_p$ and the operator $y + \overline{y}$ is carried out mod$\frac{p-1}{2}$. That is, $y + \overline{y} = y^*$ where $y^* \in \left[0, \frac{p-1}{2} - 1\right]$ and $y + \overline{y} \equiv y^*, \mathrm{mod}\frac{p-1}{2}$. Thus, $\circ$ is a closed operator on $\left\{f_{(x,y)} : x \in [0, p-1], y \in \left[0, \frac{p-1}{2} - 1\right]\right\}$.

Proof. $\left(f_{(x,y)} \circ f_{(\overline{x},\overline{y})}\right)(t) = f_{(x,y)}\left(f_{(\overline{x},\overline{y})}(t)\right) = f_{(x,y)}\left(m^{\overline{y}} \cdot t + \overline{x}\right) = m^y \cdot \left(m^{\overline{y}} \cdot t + \overline{x}\right) + x = m^{y+\overline{y}} \cdot t + (m^y \cdot \overline{x} + x) = \left(f_{(m^y \cdot \overline{x} + x, y + \overline{y})}(t)\right)$. ■

Lemma 10. $\left(\left\{f_{(x,y)} : x \in [0, p-1], y \in \left[0, \frac{p-1}{2} - 1\right]\right\}, \circ\right)$ is a group of permutations on $[0, p-1]$ where $\circ$ is the composition of functions.

Proof.

1. We prove $f_{(0,0)}$ is the identity. Note that $f_{(0,0)} \circ f_{(x,y)} = f_{(m^0 \cdot x + 0, 0 + y)} = f_{(x,y)}$. Also, $f_{(x,y)} \circ f_{(0,0)} = f_{(m^y \cdot 0 + x, y + 0)} = f_{(x,y)}$.

2. Of course, the composition of functions $(\circ)$ is always associative.

3. We show that $f_{(x,y)}$ and $f_{(-m^{-y}x, -y)}$ are inverse permutations where $-y$ is carried out mod$\frac{p-1}{2}$.

That is, $-y = y^*$ where $y^* \in [0, \frac{p-1}{2} - 1]$ and $-y \equiv y^* \mathrm{mod}\frac{p-1}{2}$. Also, once $-y$ is computed, the other operations $-m^{-y}x$ are carried out in $\mathbf{Z}_p$. Now $f_{(x,y)} \circ f_{(-m^{-y}x, -y)} = f_{(m^y \cdot (-m^{-y}x) + x, y + (-y))} = f_{(-(m^y \cdot m^{-y})x + x, 0)} = f_{(-m^{y-y} \cdot x + x, 0)} = f_{(-x+x, 0)} = f_{(0,0)}$. Therefore, $f_{(x,y)}$ and $f_{(-m^{-y}x, -y)}$ are inverse permutations. This completes the argument. ■

Let us now think about the p-star $\left(D_{[0,p-1]}, *\right)$. Using the permutations $f_{(x,y)}(t) = m^y \cdot t + x$ on $\{l_0, l_1, \cdots, l_{p-1}\} = [0, p-1]$, where $x \in [0, p-1]$, $y \in \left[0, \frac{p-1}{2} - 1\right]$, as always let $\overline{f}_{(x,y)} = \overline{f}_{(x,y)}(\{i, j\}) = \left\{f_{(x,y)}(i), f_{(x,y)}(j)\right\} = \{m^y \cdot i + x, m^y \cdot j + x\}$ be the corresponding line preserving permutations on
$\{\{l_i, l_j\} : i \neq j, i, j \in [0, p-1]\} = \{\{i, j\} : i \neq j, i, j \in [0, p-1]\} = D_{[0,p-1]}$.

Since $\overline{f \circ g} = \overline{f} \circ \overline{g}$ and $f \neq g$ implies $\overline{f} \neq \overline{g}$, we know that the line preserving permutations $\overline{f}(\{i, j\})$ on $D_{[0,p-1]}$, where $x \in [0, p-1]$, $y \in \left[0, \frac{p-1}{2} - 1\right]$, forms a group under composition of functions.

Lemma 11. The groups $\left(\left\{f_{(x,y)} : x \in [0, p-1], y \in \left[0, \frac{p-1}{2} - 1\right]\right\}, \circ\right)$ and
$\left(\left\{\overline{f}_{(x,y)} : x \in [0, p-1], y \in \left[0, \frac{p-1}{2} - 1\right], \circ\right\}\right)$ are isomorphic.

Proof. This follows from Lemmas 1,2. ■

Lemma 12. The group $\left(\left\{\overline{f}_{(x,y)} : x \in [0, p-1], y \in \left[0, \frac{p-1}{2} - 1\right]\right\}, \circ\right)$ of line preserving permutations on $D_{[0,p-1]}$ is uniquely transitive on $D_{[0,p-1]}$.

Proof. We must show that $\forall \{i, j\}, \left\{\overline{i}, \overline{j}\right\} \in D_{[0,p-1]}, \exists$ a unique ordered pair (x, y) with $x \in [0, p-1]$, $y \in \left[0, \frac{p-1}{2} - 1\right]$ such that $\left(\overline{f}_{(x,y)}\right)(\{i, j\}) = \left\{\overline{i}, \overline{j}\right\}$. That is, $\{m^y i + x, m^y j + x\} = \left\{\overline{i}, \overline{j}\right\}$ Now $D\left(\left\{\overline{i}, \overline{j}\right\}\right) = \left\{\overline{i} - \overline{j}, \overline{j} - \overline{i}\right\}$ is a doubleton subset of $[1, p-1]$ since $\overline{i} \neq \overline{j}$. Also, $D(\{m^y i + x, m^y j + x\}) = D\{m^y i, m^y j\}$. Therefore, we must have $D(\{m^y i, m^y j\}) = D\left(\left\{\overline{i}, \overline{j}\right\}\right)$.

Since $i \neq j$ from Lemma 7, the sets $D(\{m^y i, m^y j\})$, $y = 0, 1, 2, \cdots, \frac{p-1}{2} - 1$, are pairwise disjoint doubleton sets that partition $[1, p-1]$. Also, from Lemma 6, c we know that $\forall y \in \left[0, \frac{p-1}{2} - 1\right], D(\{m^y i, m^y j\}) \cap D\left(\left\{\overline{i}, \overline{j}\right\}\right) = \phi$ or $D(\{m^y i, m^y j\}) = D\left(\left\{\overline{i}, \overline{j}\right\}\right)$. Therefore, it follows that $\exists$ a unique $y \in \left[0, \frac{p-1}{2} - 1\right]$ such that $D(\{m^y i, m^y j\}) = D\left(\left\{\overline{i}, \overline{j}\right\}\right)$. Using this unique y in $[0, \frac{p-1}{2} - 1]$, by Lemma 6, (d), $\exists$ a unique $x \in [0, p-1]$ such that $\{m^y i, m^y j\} + x = \{m^y i + x, m^y j + x\} = \left\{\overline{i}, \overline{j}\right\}$. ■

10 Constructing the Group $\left(D_{[0,p-1]},\cdot\right)$

Proof of Lemma 4. Using the machinery that we developed in section 9, we now construct a group $\left(D_{[0,p-1]},\cdot\right)$ that left-distributes over the p-star $\left(D_{[0,p-1]},*\right)$ when p is a prime of the form $p=4k+3$.

We know that $\forall x\in[0,p-1]\,,\forall y\in\left[0,\frac{p-1}{2}-1\right]$, the permutation $\overline{f}_{(x,y)}\left(\{i,j\}\right)$ $=\{m^y i+x,m^y j+x\}$ on $D_{[0,p-1]}$ is a similarity mapping on the p-star $\left(D_{[0,p-1]},*\right)$ since it is a line-preserving permutation of $\left(D_{[0,p-1]},*\right)$. Also, this collection of similarity mappings on $\left(D_{[0,p-1]},*\right)$ is a uniquely transitive group of permutations on $D_{[0,p-1]}$ under composition of functions.

We are now in a position to use Theorem 3 of [5] to construct a group $\left(D_{[0,p-1]},\cdot\right)$ that left-distributes over the p-star $\left(D_{[0,p-1]},*\right)$ where $p=4k+3$ and p is prime. We have summarized this construction in the second paragraph of section 5. First, we must arbitrarily choose and then fix an element of $D_{[0,p-1]}$ to be the identity element of $\left(D_{[0,p-1]},\cdot\right)$.

It is most convenient to let $\{0,1\}$ be the identity. In the notation of Theorem 3, [5], we have $S=D_{[0,p-1]}$ and $\left(\overline{F},\circ\right)=\left(\left\{\overline{f}_{(x,y)}:x\in[0,p-1]\,,y\in\left[0,\frac{p-1}{2}-1\right]\right\},\circ\right)$.

Since $\left(\overline{F},\circ\right)$ is uniquely transitive on $S=D_{[0,p-1]}$ we know that $\forall\{i,j\}\in D_{[0,p-1]}\exists$ a unique $(x,y)\in[0,p-1]\times\left[0,\frac{p-1}{2}-1\right]$, (the product set), such that $\overline{f}_{(x,y)}\left(\{0,1\}\right)$ $=\{m^y\cdot 0+x,m^y\cdot 1+x\}=\{x,m^y+x\}=\{i,j\}$.

Therefore, let us write each $\{i,j\}\in D_{[0,p-1]}$ as $\{i,j\}=\{x,m^y+x\}=(x,y)$ where $(x,y)\in[0,p-1]\times\left[0,\frac{p-1}{2}-1\right]$. If we write each $\{i,j\}\in D_{[0,p-1]}$ in this unique way, then $\overline{F}$ is automatically indexed as required in condition b of theorem 3, [5] and which we have stated in paragraph 2 of section 4. This is because $\forall\{i,j\}=\{x,m^y+x\}\in S=D_{[0,p-1]},\overline{f}_{(x,y)}\left(\{0,1\}\right)=\{x,m^y+x\}=(x,y)$. The group $\left(D_{[0,p-1]},\cdot\right)$ with identity $\{0,1\}=\left\{0,m^0+0\right\}=(0,0)$ that left-distributes over the p-star $\left(D_{[0,p-1]},*\right)$ is now defined in theorem 3, [5] and also stated in paragraph 2 of section 5 as follows, $\forall\{x,m^y+x\},\left\{\overline{x},m^{\overline{y}}+\overline{x}\right\}\in S=D_{[0,p-1]},\{x,m^y+x\}\cdot\left\{\overline{x},m^{\overline{y}}+\overline{x}\right\}=\overline{f}_{(x,y)}\left(\left\{\overline{x},m^{\overline{y}}+\overline{x}\right\}\right)=\left\{m^y\cdot\overline{x}+x,m^y\cdot\left(m^{\overline{y}}+\overline{x}\right)+x\right\}=$ $\left\{\left(m^y\cdot\overline{x}+x\right),m^{y+\overline{y}}+\left(m^y\cdot\overline{x}+x\right)\right\}$ where as always $y+\overline{y}\in\left[0,\frac{p-1}{2}-1\right]$ is computed mod $\frac{p-1}{2}$ and the other operations are computed in the field $\mathbf{Z}_p$. Of course, $m^y\cdot\overline{x}+x\in[0,p-1]$. Note that we could also call this $(x,y)\cdot(\overline{x},\overline{y})=(m^y\cdot\overline{x}+x,y+\overline{y})$. Compare this to Lemma 9.

11 Discussion

The reader might like to compute explicitly in table form a group that left-distributes over the 7-star. Also, the groups that we have defined in this paper interact with the corresponding p-stars in many interesting ways that we do not discuss hence due to a lack of space.

Likewise, we decided not to prove that all groups that left (or right) distribute over a p-star can be constructed by the algorithm given in this paper. We do note that if $(G,\cdot)$ is a group of order $\frac{p(p-1)}{2}$ where p is an odd prime, then there exist a normal subgroup $(H,\cdot)$ of $(G,\cdot)$ such $|H|=p$. This fact follows from the Syloe theorems.

We now get the reader started on the 7-star table. For $p=7$, we use $m=2$ and we call $(x,y)=\{x,2^y+x\}=\{i,j\}$ where $x\in\{0,1,2,3,\cdots,6\}$ and $y\in[0,1,2\}$.

As always, we have $\{l_i,l_j\}=\{i,j\},i\neq j$. We can now identify the 21 points $\{i,j\}\,,i\neq j$, of the 7-star from the following table.

(x,y)	$\{x, 2^y + x\} = \{i, j\}$	(x,y)	$\{x, 2^y + x\} = \{i, j\}$
$(0,0)$	$\{0,1\}$	$(4,1)$	$\{4,6\}$
$(1,0)$	$\{1,2\}$	$(5,1)$	$\{5,0\}$
$(2,0)$	$\{2,3\}$	$(6,1)$	$\{6,1\}$
$(3,0)$	$\{3,4\}$	$(0,2)$	$\{0,4\}$
$(4,0)$	$\{4,5\}$	$(1,2)$	$\{1,5\}$
$(5,0)$	$\{5,6\}$	$(2,2)$	$\{2,6\}$
$(6,0)$	$\{6,0\}$	$(3,2)$	$\{3,0\}$
$(0,1)$	$\{0,2\}$	$(4,2)$	$\{4,1\}$
$(1,1)$	$\{1,3\}$	$(5,2)$	$\{5,2\}$
$(2,1)$	$\{2.4\}$	$(6,2)$	$\{6,3\}$
$(3,1)$	$\{3,5\}$		

As always, we multiply by using $(x,y)\cdot(\overline{x},\overline{y}) = (2^y \cdot \overline{x} + x, y + \overline{y})$ where $2^y \cdot \overline{x} + x$ is computed in Z_7 and $y + \overline{y} \in \{0,1,2\}$ is computed modulo 3. Using the table, we see that $l_5 = \{\{0,5\}, \{1,5\}, \{2,5\}, \{3,5\}, \{4,5\}, \{6,5\}\} = \{(5,1), (1,2), (5,2), (3,1), (4,0), (5,0)\}$. Also, $\{4,6\} = (4,1)$.

Therefore, $\{4,6\} \cdot l_5 = (4,1) \cdot \{(5,1), (1,2), (5,2), (3,1), (4,0), (5,0)\}$. Now $(4,1)\cdot(5,1) = (2' \cdot 5 + 4, 2) = (14,2) = (0,2) = \{0,4\}$.

Similarly $(4,1)\cdot(1,2) = \left(2^1 \cdot 1 + 4, 3\right) = (6,0) = \{0,6\}$. Also, $(4,1)\cdot(5,2) = (0,0) = \{0,1\}$, $(4,1)\cdot(3,1) = (3,2) = \{0,3\}$, $(4,1)\cdot(4,0) = (5,1) = \{0,5\}$, $(4,1)\cdot(5,0) = (0,1) = \{0,2\}$.

Thus, $\{4,6\} \cdot l_5 = l_0$ which is exactly what we expected since the group $(D_7, \cdot)$ that we are constructing on D_7 is supposed to left-distribute over the 7-star $(D_7, *)$.

12 Figures

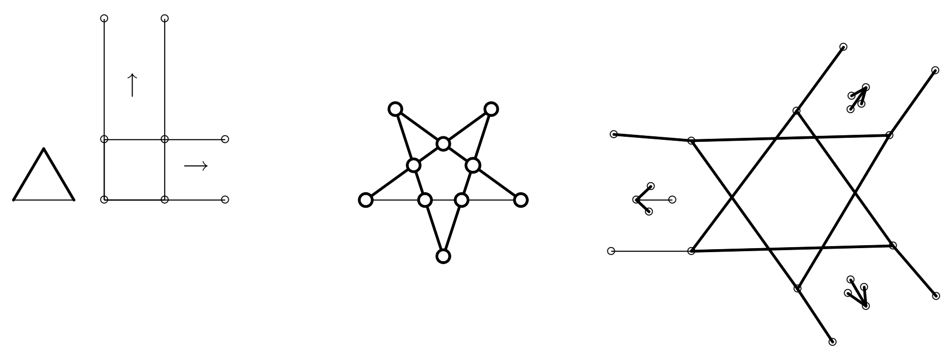


Fig. 2. Stars for $n = 3, 4, 5, 6$.

Fig. 3. A 7-star.

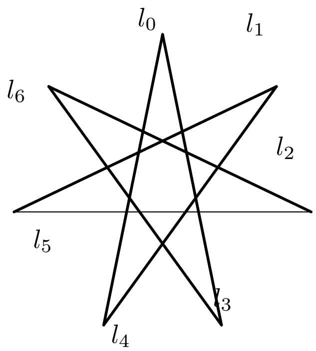
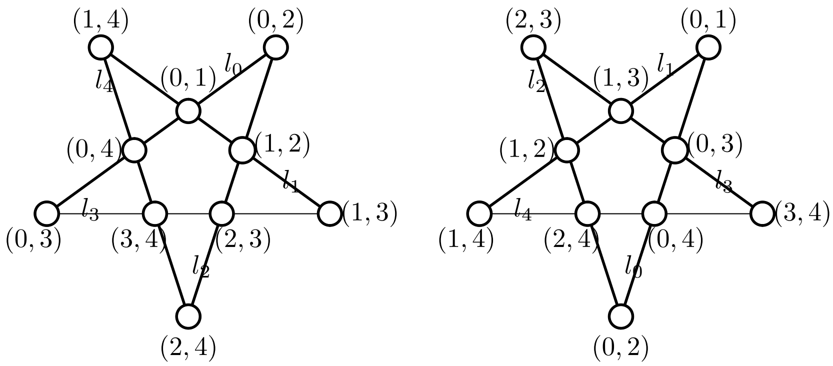


Fig. 4. A line preserving permutation.



AMS Classification: 22F50

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