A Special Case of Poncelet's Problem

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1 Abstract

A special case of Poncelet's Theorem states that if circle C_2 lies inside of circle C_1 and if a convex n-polygon, $n \geq 3$, or an n-star, $n \geq 5$, is inscribed in C_1 and circumscribed about C_2, then there exists a family of such n-polygons and n-stars. See [3] and [4].

Suppose C_2 lies inside of C_1 and R, r, are the radii of C_1, C_2 respectively and ρ is the

distance between the centers of C_1, C_2. For $n \geq 3$ we give an algorithm that computes the necessary and sufficient conditions on R, r, ρ, where $R > r + |\rho|, r > 0$, so that if we start at any arbitrary point P on C_1 and draw successive tangents to C_2 (counterclockwise about the center of C_2) then we will return to P in exactly n-steps and not sooner. This will create the above families of n-polygons and n-stars. The algorithm uses only rational operations.

2 Introduction

Jean-Victor Poncelet, born July 1, 1788, Metz, France–died December 22, 1867, Paris, was a French mathematician and engineer who was one of the founders of modern projective geometry. See [1] and [3]. As a lieutenant of engineers in 1812, he took part in Napoleon's Russian campaign, in which he was abandoned as dead at Krasnoy and imprisoned at Saratov; he returned to France in 1814. During his imprisonment Poncelet studied projective geometry and wrote Applications d´analyse et de géométrie, 2 vol. (1862–64; Applications of Analysis and Geometry).

A special case of Poncelet's Theorem states that if all points on circle C_2 lie inside of circle C_1 and if a convex n-polygon, $n \geq 3$, is inscribed in C_1 and circumscribed about C_2 then there exists a family of such n-polygons. The same thing is true when an n-star, $n \geq 5$, is inscribed in C_1 and circumscribed about C_2 and the n-star goes around the center of C_2 exactly two times or exactly three times or exactly four times, etc. Each member of the family can be constructed by starting at any arbitrary point P on C_1 and drawing successive tangents to C_2 (counterclockwise to the center of C_2) until after exactly n steps and not sooner.

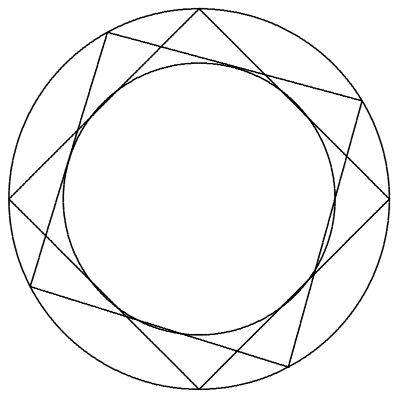

Fig. 1. A Family of Quadrilaterals.

If R, r are the radii of C_1, C_2 respectively and ρ is the distance between the centers, where $R > r + |\rho|, r > 0$, then Poncelet's Theorem and physical reasoning indicates that if $R, \rho, R > |\rho| \geq 0$, are fixed, then r must be the same and unique for all n-polygons, $n \geq 3$, of our family and for all n-stars, $n \geq 5$, of our family that go around the center of C_2 exactly two times, that go around the center of C_2 exactly three times, etc. With $R > |\rho| \geq 0$ being fixed and r being a variable, we develop a rational algorithm for computing this relation between $R, r, \rho, R > r + |\rho|, r > 0$, for all $n \geq 3$. We do this by studying a very special case for C_1, C_2, P. We assume that C_2 lies inside of C_1 and we define $C_2 : x^2 + y^2 = r^2, C_1 : (x - \rho)^2 + y^2 = R^2$. We also assume that R, ρ are fixed where $0 \leq |\rho| < R$. Then we compute the necessary and sufficient conditions on R, r, ρ where $R > r + |\rho|, r > 0$, so that if we start at $(x_0, y_0) = (r, -y_1) = \left(r, -\sqrt{R^2 - (r - \rho)^2}\right)$ and draw tangents successively to C_2 (counterclockwise about the origin $(0, 0)$) then in exactly $n \geq 3$ steps and not in fewer than n steps we will return to $(x_0, y_0) = (r, -y_1)$.

By Poncelet's Theorem these conditions are also necessary and sufficient so that if we use any arbitrary point P on C_1 in the place of $(x_0, y_0) = (r, -y_1)$ and use the same construction of tangents to C_2 (counterclockwise about $(0, 0)$) then we will return to P in exactly n-steps and never return to P in fewer than n-steps. Of course, for each fixed $n \geq 3$, this algorithm is dealing with the n-polygons and the n-stars together to generate one equation $P_n^* (R, r, \rho) = 0$ where P_n^* is a polynomial. However, for each fixed $n \geq 3$, if $R, \rho, R > |\rho| \geq 0$, are fixed and r is a variable and if the positive real r-roots of $P_n^* (R, r, \rho) = 0$ that satisfy $0 < r < R - |\rho|$ are $0 < r_1 < r_2 < \cdots < r_k < R - |\rho|$, then r_k is the radius of C_2, so that we get an n-polygon that goes around $(0, 0)$ exactly one time, r_{k-1} is the radius of C_2 so that we get an n-star that goes around $(0, 0)$ exactly two times, r_{k-2} is the radius of C_2 so that we get an n-star that goes around $(0, 0)$ exactly three times, etc. (Important: see Section 3 for a slight correction to this statement.) We call the n-polygons, $n \geq 3$ and the n-stars, $n \geq 5$, that we generate Poncelet n-polygons and Poncelet n-stars. They can also be called the standard n-gons and the standard n-stars. It may be true that $P_n^* (R, r, \rho) = 0$ has extraneous roots r_i that lie outside of $R > r_i + |\rho|, r_i > 0$. Also, $P_n^* (R, r, \rho) = 0$ might repeat some of the roots r_i. But we can eliminate this multiplicity by agreeing to write $P_n^* (R, r, \rho) = 0$ in the canonical form of comment 1.

Suppose $n \geq 3$ is fixed. In Section 3 we study exactly how many n-stars can exist and exactly how many times each n-star goes around $(0,0)$. The list $0 < r_1 < r_2 < \cdots < r_k, R > r_i + |\rho|, r_i > 0$, includes exactly one r_i for each n-star that can exist. We can see this fact intuitively by letting $R > |\rho| \geq 0$ be fixed and then letting r slowly decrease from $r = R - |\rho|$ to $r = 0$ and studying the action by using physical reasoning.

3 Initial Concepts

We first discuss what we mean by an n-star in this note. Suppose we draw a regular n-gon where $n \geq 5$ and number the vertices $1, 2, 3, \cdots, n$ in counterclockwise order. For each $k \in \left\{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\right\}$ if (n, k) are relatively prime let us start at vertex 1 and draw lines connecting $(1, 1+k), (1+k, 1+2k), (1+2k, 1+3k), (1+3k, 1+4k), \cdots$ where the calculations use modulo n arithmetic. Since (n, k) are relatively prime, we will return to vertex 1 in exactly n-steps and in no fewer than n-steps. In doing this we create an n-star that goes around the center of the n-gon exactly k times. Thus, for the 7-gon we can create 7-stars that go around the center $k = 1, k = 2$ or $k = 3$ times where we consider the 7-gon itself as a star.

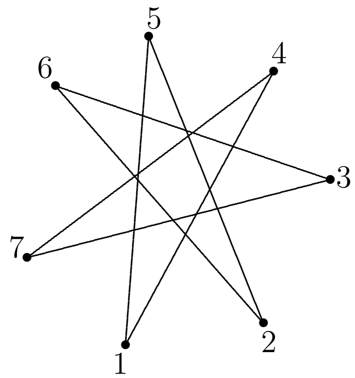

Fig. 2a. $n = 7, k = 3$

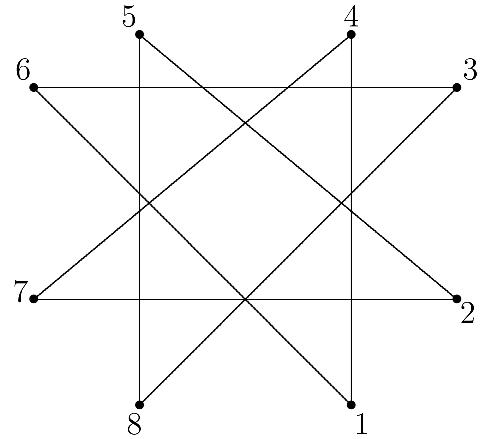

Fig. 2b. $n = 8, k = 3$

For the 8-gon we can create 8-stars that go around the center $k = 1$ or $k = 3$ times where we consider the 8-gon itself as a star.

Note 1 In this entire note, it is convenient to think of $R, \rho, R > |\rho| \geq 0$, as constants and r, where $R > r + |\rho|, r > 0$, as a variable. By doing this we can use single variable algebra and single variable calculus.

Algorithm 1. Suppose $P(R, r, \rho) = 0$, $Q(R, r, \rho) = 0$ are two polynomial equations (where r is the variable) and we wish to eliminate all r-variable traces of $P(R, r, \rho) = 0$ that are embedded in $Q(R, r, \rho) = 0$ and leave the rest. The following algorithm does this and it also explains exactly what we mean. (In comment 1 we mention possible overkill.)

1. First, compute $Q_1 = \gcd(P, Q)$ and write $Q = Q_1 \cdot Q'$ where gcd denotes greatest common divisor and Q_1 is a polynomial in R, r, ρ. All calculations consider r the variable.

2. Next, compute $Q_2 = \gcd(P, Q')$ where Q_2 is a polynomial in R, r, ρ and write $Q^{'} = Q_2 \cdot Q''$ so that $Q = Q_1 Q_2 Q''$.

3. Next, compute $Q_3 = \gcd\left(P, Q''\right)$ where Q_3 is a polynomial and write $Q'' = Q_3 \cdot Q'''$ so

that $Q = Q_1 Q_2 Q_3 Q'''$.

$\vdots$

n. Last, compute $Q_n = \gcd\left(P, Q^{(n-1)}\right)$ where Q_n is a polynomial and write $Q^{(n-1)} = Q_n \cdot Q^{(n)}$ so that $Q = Q_1 \cdot Q_2 \cdots Q_n \cdot Q^{(n)}$. Suppose now that $\gcd\left(P, Q^{(n)}\right) = 1$. That is, $P, Q^{(n)}$ are relatively prime in the variable r. We now define $Q^{(n)}$ to be the part (or divisor) of Q that remains after we eliminate all traces of P in the variable r that are embedded in Q.

Since we will always be writing the equation $Q^{(n)} = 0$, we can also write $Q^{(n)}$ as a polynomial in all of the variables R, r, ρ.

If we wish to eliminate all r-variable traces of several polynomials $P_1(R, r, \rho) = 0, P_2(R, r, \rho) = 0, \cdots, P_k(R, r, \rho) = 0$ that are embedded in polynomial $Q(R, r, \rho) = 0$ and leave the rest we first use the above algorithm with (P_1, Q). Let Q^* be the divisor of Q that remains after all r-traces of P_1 have been eliminated from Q.

We next use the algorithm with (P_2, Q^*), and let Q^{**} be the divisor of Q^* that remains after all r-traces of P_2 have been removed from Q^*. Then we use the algorithm with (P_3, Q^{**}) and let Q^{***} be the divisor of Q^{**} that remains after all r-traces of P_3 have been eliminated from Q^{**} .

We continue the algorithm with each $P_1, P_2 \cdots P_k$ until we end up with $Q^{***\cdots*}$ where $Q^{***\cdots*}$ is the divisor of Q that remains after all r-traces of $P_1, P_2, \cdots, P_k$ have been eliminated from Q.

<u>Comment 1</u> In applying this algorithm to the problems in this note, from our experience we believe that to eliminate all r-traces of a polynomial $P(R, r, \rho) = 0$ from a polynomial

$Q(R,r,\rho)=0$, then all we have to do is divide $\frac{Q(R,r,\rho)}{P(R,r,\rho)}=Q'=Q^{(n)}$ one time and $Q'=Q^{(n)}$ will automatically be the answer that we are seeking. However, only more practice will tell us whether this is always true or not. As always, we let R,ρ be fixed and r be a variable. Suppose a polynomial $P(R,r,\rho)=0$ in the rational field is factored $P(R,r,\rho)=P_1^{k_1}(R,r,\rho)\cdot P_2^{k_2}(R,r,\rho)\ \cdots P_n^{k_n}(R,r,\rho)$ where $P_1,P_2,\cdots P_n$ are distinct polynomials (in the rational field) in the variable r that are each irreducible in the rational field. Then by algebra and calculus we can compute a polynomial $\overline{P}(R,r,a)=\dfrac{P(R,r,\rho)}{\gcd(D_rP,P)}=P_1(R,r,\rho)\cdot P_2(R,r,\rho)\cdots P_n(R,r,\rho)$. D_r is the r-variable derivative. We call $\overline{P}(R,r,\rho)$ the canonical form of $P(R,r,\rho)$. This polynomial $\overline{P}(R,r,\rho)=0$ will contain the exact same r-root information as $P(R,r,\rho)=0$ since they have the same r-roots but $\overline{P}(R,r,\rho)$ does not repeat the r-roots. $\overline{P}(R,r,\rho)$ is all that we need. We do not have to compute $P_1,P_2,\ldots,P_k$.

If we agree to write all of our polynomials in this canonical form $\overline{P}(R,r,\rho)$, then in this note it becomes much more likely that Algorithm 1 can be carried out by the above single division $\frac{Q(R,r,\rho)}{P(R,r,\rho)}=Q'=Q^{(n)}$. In any case, if we write all of our polynomials in the above canonical form, Algorithm 1 can always be carried out in just one single step.

4 Analytic Machinery

As always, in this note $C_2: x^2+y^2=r^2, C_1: (x-\rho)^2+y^2=R^2$ are the standard definitions of two circles and C_2 lies inside of C_1. That is, $R>r+|\rho|, r>0$. The origin $(0,0)$ is the center of C_2 and $(\rho,0)$ is the center of C_1.

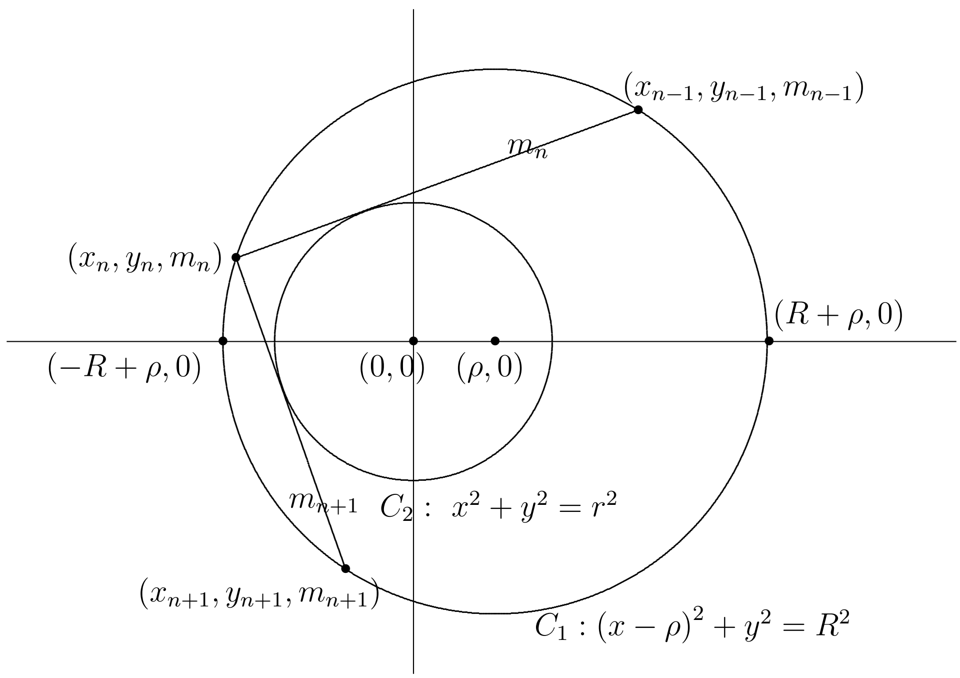

Fig. 3 The Standard Circles, $C_2 : \ x^2 + y^2 = r^2$ and $C_1 : (x-\rho)^2 + y^2 = R^2$

In Fig. 3 and throughout this paper, the reader may prefer to let $\rho \geq 0$. Suppose $(x_{n-1}, y_{n-1}, m_{n-1}), (x_n, y_n, m_n), (x_{n+1}, y_{n+1}, m_{n+1})$ are drawn in Fig. 3, and suppose that $(x_{n-1}, y_{n-1}), (x_n, y_n), (x_{n+1}, y_{n+1})$ are successive points on circle C_1 and the tangent lines to circle C_2 in Fig. 3 are oriented counterclockwise about the origin $(0,0)$ as indicated by the arrows. Also, $m_n, m_{n+1}, \cdots$ are the reciprocals of the slopes of the tangent lines in Fig. 3. That is, $m_n = \frac{x_n - x_{n-1}}{y_n - y_{n-1}}, m_{n+1} = \frac{x_{n+1} - x_n}{y_{n+1} - y_n}, \cdots$.

For each successive $n, n+1$, the line between (x_n, y_n) and (x_{n+1}, y_{n+1}) can be defined parametrically by the equation $(x, y) = (x_n + m_{n+1}t, y_n + t)$ where $t \in R$ is the parameter.

Using the elementary analytic geometry of the circle, we can easily derive the following recursive equations $(x_n, y_n, m_n) \to (x_{n+1}, y_{n+1}, m_{n+1})$ where (x_0, y_0, m_0) is a given starting point.

1. $m_{n+1} = \dfrac{2x_n y_n}{y_n^2 - r^2} - m_n.$
2. $x_{n+1} = \dfrac{(-x_n + 2\rho)\, m_{n+1}^2 - 2y_n m_{n+1} + x_n}{m_{n+1}^2 + 1}.$
3. $y_{n+1} = \dfrac{y_n m_{n+1}^2 - 2\,(x_n - \rho)\, m_{n+1} - y_n}{m_{n+1}^2 + 1}.$

5 A Special Case of the Recursion

As stated previously, this standard special case if the case that we always deal with in this note. Suppose we define $(x_0, y_0, m_0) = \left(r, -\sqrt{R^2 - (r-\rho)^2}, m_0\right)$ and $(x_1, y_1, m_1) = \left(r_1, \sqrt{R^2 - (r-\rho)^2}, 0\right)$ where the line between $(x_0, y_0), (x_1, y_1)$ is a vertical tangent to circle C_2.

We note that $x_1 = r$ is a rational function of r. Also, we note that y_1 is an irrational function of R, r, ρ but y_1^2 is a rational function of R, r, ρ.

By studying the recursive equations of Section 4, we easily see by using induction that we can write $(x_n, y_n, m_n) = (x_n, Y_n \cdot y_1, M_n \cdot y_1)$ where x_n, Y_n, M_n are rational functions of R, r, ρ and $y_1 = +\sqrt{R^2 - (r-\rho)^2}$. Since $y_1^2 = R^2 - (r-\rho)^2$ is a rational function of R, r, ρ, it follows by induction from $(x_n, y_n, m_n) = (x_n, Y_n \cdot y_1, M_n \cdot y_1)$ and from the recursive equations of Section 4 that $(x_{n+1}, y_{n+1}, m_{n+1}) = (x_{n+1}, Y_{n+1} \cdot y_1, M_{n+1} \cdot y_1)$ where $x_{n+1}, Y_{n+1}, M_{n+1}$ are rational functions of R, r, ρ.

If $(x_0, y_0, m_0) = \left(r, -\sqrt{R^2 - (r-\rho)^2}, m_0\right)$ and $(x_1, y_1, m_1) = \left(r, +\sqrt{R^2 - (r-\rho)^2}, 0\right)$, we see that the recursive equations $1, 2, 3$ of Section 4 can now be written for x_n, Y_n, M_n as the following recursion where, of course, $(x_n, y_n, m_n) = (x_n, Y_n \cdot y_1, M_n \cdot y_1)$ and $(x_{n+1}, y_{n+1}, m_{n+1}) = (x_{n+1}, Y_{n+1} \cdot y_1, M_{n+1} \cdot y_1)$ and where we use $y_1^2 = R^2 - (r-\rho)^2$ and $R > r + |\rho|, r > 0$.

1. $(x_1, Y_1, M_1) = (r, 1, 0)$
2. $M_{n+1} = \dfrac{2x_n Y_n}{Y_n^2 \cdot y_1^2 - r^2} - M_n.$
3. $x_{n+1} = \dfrac{(-x_n + 2\rho)\, M_{n+1}^2 \cdot y_1^2 - 2Y_n M_{n+1} \cdot y_1^2 + x_n}{M_{n+1}^2 \cdot y_1^2 + 1}.$
4. $Y_{n+1} = \dfrac{Y_n M_{n+1}^2 \cdot y_1^2 - 2\,(x_n - \rho)\, M_{n+1} - Y_n}{M_{n+1}^2 \cdot y_1^2 + 1}.$

In these equations, we let R, ρ be constants and let r be the variable. We can even let $R = 1$. There is also no loss of generality if we assume $\rho \geq 0$.

The computer programs run more efficiently if we deal exclusively with polynomials. Therefore, let us write $M_n = \frac{M_n}{\overline{M}_n}, x_n = \frac{x_n}{\overline{x}_n}, Y_n = \frac{Y_n}{\overline{x}_n}$ where we have the five polynomials, $x_n, \overline{x}_n, Y_n, M_n, \overline{M}_n$.

We now have $x_1 = r, \overline{x}_1 = 1, Y_1 = 1, M_1 = 0, \overline{M}_1 = 1, y_1^2 = R^2 - (r-\rho)^2$.

The recursions are as follows.

1. $M_{n+1} = 2x_n Y_n \overline{M}_n - Y_n^2 M_n y_1^2 + r^2 \overline{x}_n^2 M_n.$

$1'$ $\overline{M}_{n+1} = Y_n^2 \overline{M}_n y_1^2 - r^2 \overline{x}_n^2 \overline{M}_n.$

2. $x_{n+1} = (-x_n + 2\rho\overline{x}_n)\, M_{n+1}^2 y_1^2 - 2Y_n M_{n+1} \overline{M}_{n+1} y_1^2 + x_n \overline{M}_{n+1}^2.$

$2'$ $\overline{x}_{n+1} = \overline{x}_n M_{n+1}^2 y_1^2 + \overline{x}_n \overline{M}_{n+1}^2.$

3. $Y_{n+1} = Y_n M_{n+1}^2 y_1^2 - 2\left(x_n - \rho \overline{x}_n\right) M_{n+1} \overline{M}_{n+1} - Y_n \overline{M}_{n+1}^2$.

We can easily prove by induction that for all $n \geq 1$ and for all real R, r, ρ we have $\overline{x}_n > 0, \overline{x}_{n+1} > 0$.

Therefore, we never have to worry about $\frac{x_n}{\overline{x}_n}, \frac{Y_n}{\overline{x}_n}$ having a common r-root in the range $R > r + |\rho|, r > 0$.

However, to be on the safe side we need to compute the gcd $\left(M_n, \overline{M}_n\right)$ and throw this gcd away, in the numerator and denominator of $\frac{M_n}{\overline{M}_n}$.

In this note, we always deal with the fraction form of the recursion and not the polynomial form.

In both the fraction and polynomial forms of the recursion, it appears that the recursive equations will quickly become intractable. However, from our experience, these recursive equations will massively simplify proportional to the expansion. So they remain tractable. This phenomenon is far from random.

<u>Comment 2</u> It is probably true by induction that $y_1^2 | \left(x_n - r\right)$ for all $n \in \{0, 1, 2, 3, \cdots\}$ where $y_1^2 = \left(R - r + \rho\right)\left(R + r - \rho\right)$.

To see this we see that $x_0 = x_1 = r$ and $y_1^2 | \left(x_0 - r\right)$ and $y_1^2 | \left(x_1 - r\right)$. From the fraction form of the recursion for x_{n+1} we see that $x_{n+1} - r = \frac{(\quad) y_1^2 + (x_n - r)}{M_{n+1}^2 y_1^2 + 1}$ and from this we see that it is probably true that $y_1^2 | \left(x_{n+1} - r\right)$ since $y_1^2 | y_1^2$ and $y_1^2 | \left(x_n - r\right)$.

By the same reasoning it is also probably true by induction that $r \,| x_n$ and $r | \, M_n$ for all $n \in \{1, 2, 3, \cdots\}$. To see this we see that $r \,| x_1, r | \, M_1$ since $x_1 = r, M_1 = 0$.

From the recursion for x_{n+1}, M_{n+1}, we see that it is probably true that $r | x_{n+1}, r | M_{n+1}$ for all $n \in \{1, 2, 3, \cdots\}$.

6 Main Problem 1 and Problems $1, 1', 2$

Main Problem 1 Suppose $n \geq 3$ is fixed. Using the standard example that we defined in Section 5, where $R, \rho, R > |\rho| \geq 0$, are fixed, we wish to compute the necessary and sufficient conditions $P_n^*(R, r, \rho) = 0, R > r + |\rho|, r > 0$, where r is considered to be the only variable and where $P_n^*(R, r, \rho)$ is a polynomial in R, r, ρ, so that if we start at the standard point $(x_0, y_0) = (r, -y_1) = \left(r, -\sqrt{R^2 - (r-\rho)^2}\right)$ on C_1 and draw successive tangents to C_2 that are oriented counterclockwise about $(0,0)$ then we will return to (x_0, y_0) in exactly n steps and also such that we pass through (x_0, y_0) just one time in n steps. (In this note, when we say that we arrive at or return to (x_0, y_0) in exactly n steps this always means that we arrive at or return to (x_0, y_0) at the end of exactly n steps.)

Note 2 We will call this $P_n^*(R, r, \rho) = 0$, where $R > r + |\rho|, r > 0$ the standard equation or the Poncelet equation. As always, starting at (x_0, y_0), we call the above construction of tangents to C_2 the standard construction and we call $(x_0, y_0) = \left(r, -\sqrt{R^2 - (r-\rho)^2}\right) \rightarrow (x_1, y_1) = \left(r, +\sqrt{R^2 - (r-\rho)^2}\right)$ the standard starting points.

Observation 1 We soon define three problems whose solutions are equivalent to Main Problem 1. First, we state the following without proof. By the x-axis symmetry of the standard construction, the proofs are fairly easy and are left to the reader.

Suppose we start at the standard $(x_0, y_0) = (r, -y_1)$ and by using the standard construction we arrive back at (x_0, y_0) in exactly n steps and also we arrive back at (x_0, y_0) just one time in n steps. We call this the standard condition (or the Poncelet condition).

1. If $n \geq 3$ is odd, then the standard (or Poncelet) condition is met if and only if in exactly $\frac{n+1}{2}$ steps we arrive at one of the two points $(-R + \rho, 0), (R + \rho, 0)$.

Also, we pass through this $(-R+\rho,0)$ or $(R+\rho,0)$ point exactly one time in $\frac{n+1}{2}$ steps. Note that we only pass through one of these two points $(-R+\rho,0)\,,(R+\rho,0)\,.$ Exactly which of these two points we arrive at in $\frac{n+1}{2}$ steps depends exactly upon the nature of the n-star that we are dealing with. Of course, a Poncelet n-polygon will arrive at$(-R+\rho,0)$ in exactly $\frac{n+1}{2}$ steps. The reader can study analogies of Fig. 2 to see this. When $n=3$, we have no 3-stars and we can only arrive at $(-R+\rho,0)$ in exactly $\frac{n+1}{2}=2$ steps. We cannot arrive at $(R+\rho,0)$ in 2 steps. When $n\geq 5$ is odd, we can have some n-stars (and one n-polygon) that arrive at $(-R+\rho,0)$ in exactly $\frac{n+1}{2}$ steps, and just one time in $\frac{n+1}{2}$ steps, and we can have some n-stars that arrive at $(R+\rho,0)$ in exactly $\frac{n+1}{2}$ steps and just one time in $\frac{n+1}{2}$ steps.

2. If $n\geq 4$ and n is even, then the standard (or Poncelet) condition is met if and only if in exactly $\frac{n}{2}$ steps and just one time in $\frac{n}{2}$ steps, we arrive at $\left(-r,+\sqrt{R^2-(-r-\rho)^2}\right)=\left(-r,+\sqrt{R^2-(r+\rho)^2}\right).$

   We note that there are no n-stars when $r=4$ or $n=6$. Look at the analogy of Fig. 2 for $n=6$.

From Observation 1, Main Problem 1 is equivalent to the following Problems $1,1',2$.

In Problems $1,1',2$ as always we consider R and $\rho, R>|\rho|\geq 0$, to be fixed and r to be a variable where $R>r+|\rho|\,,r>0$.

<u>Problem 1</u> Suppose $n\geq 3$ and n is odd. We wish to find necessary and sufficient conditions $P_n(R,r,\rho)=0$, where $P_n(R,r,\rho)$ is a polynomial in R,r,ρ and $R>r+|\rho|\,,r>0$, so that if we start at the standard (x_0,y_0) and use the standard construction then we will arrive at $(-R+\rho,0)$ in exactly $\frac{n+1}{2}$ steps and we also pass through $(-R+\rho,0)$

just one time in $\frac{n+1}{2}$ steps.

<u>Problem $1'$</u> Suppose $n \geq 5$ and n is odd. We wish to find necessary and sufficient condition $\overline{P}_n(R,r,\rho) = 0$, where $\overline{P}_n(R,r,\rho)$ is a polynomial in R, r, ρ and $R > r + |\rho|, r > 0$, so that if we start at the standard (x_0, y_0) and use the standard construction then we will arrive at $(R+\rho, 0)$ in exactly $\frac{n+1}{2}$ steps and we also pass through $(R+\rho, 0)$ just one time in $\frac{n+1}{2}$ steps.

The solution to Main Problem 1 when $n \geq 3$ and n is odd is $P_n^* = P_n(R,r,\rho) = 0$ or $P_n^* = \overline{P}_n(R,r,\rho) = 0$ where $R > r + |\rho|, r > 0$.

Problem $1'$ is degenerate with no solution with $R > r + |\rho|, r > 0$, when $n = 3$.

<u>Problem 2</u> Suppose $n \geq 4$ and n is even. We wish to find necessary and sufficient conditions $P_n(R,r,\rho) = 0$ where $P_n(R,r,\rho)$ is a polynomial in R, r, ρ and $R > r + |\rho|, r > 0$, so that if we start at the standard (x_0, y_0) and use the standard construction then we will arrive at $\left(-r, +\sqrt{R^2 - (-r-\rho)^2}\right) = \left(-r, +\sqrt{R^2 - (r+\rho)^2}\right)$ in exactly $\frac{n}{2}$ steps and we also pass through $\left(-r, \sqrt{R^2 - (r+\rho)^2}\right)$ just one time in $\frac{n}{2}$ steps.

The solution to Main Problem 1 when $n \geq 4$ and n is even is $P_n^*(R,r,\rho) = P_n(R,r,\rho) = 0, R > r + |\rho|, r > 0$.

7 Weaker Conditions on R, r, ρ

To solve Problems 1, $1'$, 2 we first compute some weaker conditions on R, r, ρ where $R > r + |\rho|, r > 0$.

In this section, we start at the standard $(x_0, y_0) \to (x_1, y_1, m_1)$ and we assume that we have computed $(x_n, y_n, m_n) = (x_n, Y_n \cdot y_1, M_n \cdot y_1)$ for each $n \in \{1, 2, 3, \cdots\}$ by the recursive algorithm of Section 5.

<u>Problem 1*</u> Suppose $n \geq 3$ and n is odd. We wish to find necessary and sufficient conditions $R_n(R, r, \rho) = 0$ where $R_n(R, r, \rho)$ is a polynomial in R, r, ρ and $R > r + |\rho|, r > 0$, so that if we start at the standard (x_0, y_0) and use the standard construction of tangents to C_2 then we will arrive at $(-R + \rho, 0)$ in exactly $\frac{n+1}{2}$ steps. In Problem 1* we do not require that we also arrive at $(-R + \rho, 0)$ just one time in $\frac{n+1}{2}$ steps.

<u>Problem 1**</u> Suppose $n \geq 5$ and n is odd. We wish to find necessary and sufficient conditions $\overline{R}_n(R, r, \rho) = 0$ where $\overline{R}_n(R, r, \rho)$ is a polynomial in R, r, ρ and $R > r + |\rho|, r > 0$, so that if we start at the standard (x_0, y_0) and use the standard construction of tangents to C_2 then we will arrive at $(R + \rho, 0)$ in exactly $\frac{n+1}{2}$ steps. In Problem 1** we do not require that we also arrive at $(R + \rho, 0)$ just one time in $\frac{n+1}{2}$ steps. Problem 1** has no solution when $n = 3$ that satisfies $R > r + |\rho|, r > 0$.

<u>Solution to Problems 1*, 1**</u> Problems $1^*, 1^{**}$ can be solved by settling $x_{\frac{n+1}{2}} = -R + \rho$ and $x_{\frac{n+1}{2}} = R + \rho$ respectively.

This gives the required polynomials $R_n(R, r, \rho) = 0$ and $\overline{R}_n(R, r, \rho) = 0$ where we require $R > r + |\rho|, r > 0$.

These two equations are equivalent to $x_{\frac{n+1}{2}} - r = -R + \rho - r = -(R + r - \rho)$ and $x_{\frac{n+1}{2}} - r = R - r + \rho$ respectively.

Since $y_1^2 = (R + r - \rho)(R - r + \rho)$ probably divides $x_{\frac{n+1}{2}} - r$, we see that we can probably divide out $R + r - \rho$ and $R - r + \rho$ respectively in these two equations. We can now call these new polynomials $R_n(R, r, \rho) = 0, \overline{R}_n(R, r, \rho) = 0$ and as always, we can write $R_n, \overline{R}_n$ in the canonical form. These factors $R + r - \rho = 0, R - r + \rho = 0$ are extraneous since we soon show that they each contradict $R > r + |\rho|, r > 0$. We rarely use the above solutions. The following second solutions are much superior. From Fig. 3, we can solve problem 1^* by using the equality $\frac{x_{\frac{n-1}{2}} - (-R+\rho)}{y_{\frac{n-1}{2}} - 0} = m_{\frac{n+1}{2}}$. That is, $x_{\frac{n-1}{2}} + R - \rho = y_{\frac{n-1}{2}} m_{\frac{n+1}{2}} = Y_{\frac{n-1}{2}} \ M_{\frac{n+1}{2}} y_1^2$. This is equivalent to $(*)$.

$(*) \left(x_{\frac{n-1}{2}} - r\right) + (R + r - \rho) = Y_{\frac{n-1}{2}} \ M_{\frac{n+1}{2}} y_1^2$.

Since $y_1^2 | \left(x_{\frac{n-1}{2}} - r\right)$ is probably true, we see that $R + r - \rho$ will probably divide out of $(*)$. $R + r - \rho = 0$ is extraneous since $R + r - \rho = 0, r > 0$ implies $\rho = |\rho| = R + r$ and $R \not> r + |\rho| = R + 2r$. After we divide $R + r - \rho$ out of $(*)$, we call the resulting polynomial equation $R'_n(R, r, \rho) = 0$.

Now $R'_n(R, r, \rho) = 0$ is not the solution to Problem 1^*. We now observe that $R_{n-2}(R, r, \rho) = 0$ gives necessary and sufficient conditions so that the standard construction arrives at $(-R + \rho, 0)$ in exactly $\frac{(n-2)+1}{2} = \frac{n-1}{2}$ steps, and this will also solve the above equation $(*)$ since $x_{\frac{n-1}{2}} + R - \rho = 0$ and $Y_{\frac{n-1}{2}} = 0$. Therefore, to compute the true solution to Problem 1^*, we must now eliminate all r-traces of $R_{n-2}(R, r, \rho) = 0$ from the equation $R'_n(R, r, \rho) = 0$ by using Algorithm 1 with emphasis on Comment 1. The divisor of $R'_n(R, r, \rho)$ that is left will be the true necessary and sufficient conditions $R_n(R, r, \rho) = 0, R > r + |\rho|, r > 0$, that solve Problem 1^*.

From Fig. 3, we can solve Problem 1^{**} by using the equality $\frac{x_{\frac{n-1}{2}} - (R+\rho)}{y_{\frac{n-1}{2}} - 0} = m_{\frac{n+1}{2}}$. That is, $x_{\frac{n-1}{2}} - R - \rho = y_{\frac{n-1}{2}} m_{\frac{n+1}{2}} = Y_{\frac{n-1}{2}} M_{\frac{n+1}{2}} y_1^2$. This is equivalent to $(**)$.

$(**) \left(x_{\frac{n-1}{2}} - r\right) - (R - r + \rho) = Y_{\frac{n-1}{2}} M_{\frac{n+1}{2}} y_1^2$.

Since $y_1^2 | \left(x_{\frac{n-1}{2}} - r\right)$ is probably true, we see that $R - r + \rho$ will probably divide out of $(**)$.

Now $R - r + \rho = 0$ is extraneous since $R - r + \rho = 0$ implies $\rho = -(R - r)$ which implies

$|\rho| = R - r$ and $R \not> r + |\rho| = R$.

After we divide $R - r + \rho$ out of $(**)$, we call the resulting polynomial equation $\overline{R}_n' (R, r, \rho) = 0$.

Now $\overline{R}_n' (R, r, \rho) = 0$ is not the solution to Problem 1^{**}. We now observe that $\overline{R}_{n-2} (R, r, \rho) = 0$ gives necessary and sufficient conditions so that the standard construction arrives at $(R + \rho, 0)$ in exactly $\frac{(n-2)+1}{2} = \frac{n-1}{2}$ steps and this will also solve the above equation $(**)$ since $x_{\frac{n-1}{2}} - R - \rho = 0$ and $Y_{\frac{n-1}{2}} = 0$.

Therefore, to compute the true solution to Problem 1^{**}, we must now eliminate all r-traces of $\overline{R}_{n-2} (R, r, \rho) = 0$ from the equation $\overline{R}_n' (R, r, \rho) = 0$ by using Algorithm 1 with emphasis on Comment 1. The divisor of $\overline{R}_n' (R, r, \rho)$ that is left will be the true necessary and sufficient conditions $\overline{R}_n (R, r, \rho) = 0, R > r + |\rho|, r > 0$, that solve Problem 1^{**}. ■

<u>Problem 2^*</u> Suppose $n \geq 4$ and n is even. We wish to find necessary and sufficient conditions $R_n (R, r, \rho) = 0$ where $R_n (R, r, \rho)$ is a polynomial in R, r, ρ and $R > r + |\rho|, r > 0$, so that if we start at the standard (x_0, y_0) and use the standard construction of tangents to C_2 then we will arrive at $\left(-r, +\sqrt{R^2 - (-r-\rho)^2}\right) = \left(-r, +\sqrt{R^2 - (r+\rho)^2}\right)$ in exactly $\frac{n}{2}$ steps.

In Problem 2^*, we do not require that we also arrive at $\left(-r, +\sqrt{R^2 - (-r-\rho)^2}\right)$ just one time in $\frac{n}{2}$ steps.

<u>Solution to Problem 2^*</u> We first define the equation $x_{\frac{n}{2}} = -r$ where $x_1, x_2, x_3, \cdots$ have been recursively computed. From Comment 2, we know that $r | x_{\frac{n}{2}}$ is probably true.

Therefore, we divide r out of the equation $x_{\frac{n}{2}} = -r$ where $r = 0$ is an extraneous factor since it contradicts $r > 0$. This defines a polynomial equation $R_n' (R, r, \rho) = 0, R > r + |\rho|, r > 0$ which gives necessary and sufficient conditions so that if we start at (x_0, y_0)

and construct tangents to C_2 in the standard way then we will arrive at one or the other of $\left(-r,-\sqrt{R^2-(-r-\rho)^2}\right),\left(-r,+\sqrt{R^2-(-r-\rho)^2}\right)$ in exactly $\frac{n}{2}$ steps (but not necessarily just one time in $\frac{n}{2}$ steps).

By induction, we know that $R_{n-2}=0,R>r+|\rho|,r>0$, are the necessary and sufficient conditions so that if we start at the standard (x_0,y_0) and use the standard construction of tangents to C_2, then we will arrive at $\left(-r,+\sqrt{R^2-(-r-\rho)^2}\right)$ in exactly $\frac{n-2}{2}=\frac{n}{2}-1$ steps (but not necessarily just one time in $\frac{n}{2}-1$ steps).

Now if the standard and construction starting at (x_0,y_0) arrives at $\left(-r,-\sqrt{R^2-(-r-\rho)^2}\right)$ in exactly $\frac{n}{2}$ steps, then this construction must also arrive at $\left(-r,+\sqrt{R^2-(-r-\rho)^2}\right)$ in exactly $\frac{n}{2}-1$ steps. Therefore, if we eliminate all r-traces of $R_{n-2}(R,r,\rho)=0,R>r+|\rho|,r>0$, from $R_n'(R,r,\rho)=0$ by using Algorithm 1 with emphasis on Comment 1, the divisor of $R_n'(R,r,\rho)=0$ that is left will be the necessary and sufficient conditions $R_n(R,r,\rho)=0,R>r+|\rho|,r>0$, that solves Problem 2*. As always, we can write R_n in the canonical form. If we write R_n' and R_{n-2} in the canonical form, it may be true that we only need to divide $\frac{R_n'(R,r,\rho)}{R_{n-2}(R,r,\rho)}=R_n(R,r,\rho)$.

In any case, if we write R_n' and R_{n-2} in the canonical form, then Algorithm 1 can be carried out in only one step.

8 Solving Problem 1, 1′, 2 and Main Problem 1

Notation 1 We now review the notation. As in Section 7, for each $n\geq 3,n$ odd, $R_n(R,r,\rho)=0,R>r+|\rho|,r>0$, are the necessary and sufficient conditions calculated in Section 7 so that the standard construction starting at the standard $(x_0,y_0)=(r,-y_1)$ arrives at $(-R+\rho,0)$

in exactly $\frac{n+1}{2}$ steps but not necessarily just one time in $\frac{n+1}{2}$ steps.

Also, $\overline{R}_n(R,r,\rho) = 0, R > r + |\rho|, r > 0$, are the necessary and sufficient conditions calculated in Section 7 so that the standard construction starting at the standard $(x_0, y_0) = (r, -y_1)$ arrives at $(R + \rho, 0)$ in exactly $\frac{n+1}{2}$ steps, but not necessarily just one time in $\frac{n+1}{2}$ steps.

Also, $P_n(R,r,\rho) = 0, R > r + |\rho|, r > 0$, and $\overline{P}_n(R,r,\rho) = 0, R > r + |\rho|, r > 0$, are the necessary and sufficient conditions so that the standard construction starting at the standard $(x_0, y_0) = (r, -y_1)$ arrives at $(-R + \rho, 0), (R + \rho, 0)$ respectively in exactly $\frac{n+1}{2}$ steps and passes through $(-R + \rho, 0), (R + \rho, 0)$ just one time in $\frac{n+1}{2}$ steps.

For each $n \geq 4, n$ even, $R_n(R,r,\rho) = 0, R > r + |\rho|, r > 0$, are the necessary and sufficient conditions calculated in Section 7 so that the standard construction starting at the standard $(x_0, y_0) = (-r, -y_1)$ arrives at $\left(-r, +\sqrt{R^2 - (-r - \rho)^2}\right)$ in exactly $\frac{n}{2}$ steps but not necessarily just one time in $\frac{n}{2}$ steps.

Also, for each $n \geq 4, n$ even, $P_n(R,r,\rho) = 0, R > r + |\rho|,\ r > 0$, are the necessary and sufficient conditions so that the standard construction starting at $(x_0, y_0) = (r, -y_1)$ arrives at $\left(-r, +\sqrt{R^2 - (-r - \rho)^2}\right)$ in exactly $\frac{n}{2}$ steps and passes through $\left(-r, +\sqrt{R^2 - (-r - \rho)^2}\right)$ just one time in $\frac{n}{2}$ steps.

<u>Solution to Problems 1,1′</u> Suppose $n \geq 3, n$ is odd, is fixed, and the Problems 1,1′ have been solved for all $3 \leq \overline{n} < n$ where $\overline{n}$ is odd. We wish to calculate $P_n(R,r,\rho) = 0, R > r + |\rho|, r > 0$. The calculation of $\overline{P}_n(R,r,\rho) = 0, R > r + |\rho|, r > 0$ in Problem 1′ is exactly the same as the calculation of $P_n(R,r,\rho) = 0, R > r + |\rho|, r > 0$, in Problem 1.

Suppose $3 \leq n_1 < n_2 < \cdots < n_k < n$ is the list of all positive odd integers $\overline{n}$ that lie in $3 \leq \overline{n} < n$ with the following propriety $(*)$.

Of course, as always, for each n_i in the list, $P_{n_i}(R,r,\rho)=0, R>r+|\rho|, r>0$, are the necessary and sufficient conditions so that a (Poncelet) n_i-gon or n_i-star constructed by the standard construction starting at $(x_0,y_0)=(-r,-y_1)$ arrives at $(-R+\rho,0)$ in exactly $\frac{n_i+1}{2}$ steps and passes through $(-R+\rho,0)$ just one time in $\frac{n_i+1}{2}$ steps.

<u>Property $(*)$</u> For each n_i in the list, we require these (Poncelet) n_i-gons or n_i-stars to also arrive at $(-R+\rho,0)$ in exactly $\frac{n+1}{2}$ steps.

In this note, for each odd $3\leq n$, we compute the above list $3\leq n_1<n_2<\cdots<n_k<n$ of odd $n_i's$ adhoc by simply checking each odd $3\leq\overline{n}<n$ to see if $\overline{n}$ has property $(*)$.

For our fixed $n\geq 3, n$ odd, $R_n(R,r,\rho)=0, R>r+|\rho|, r>0$ are the necessary and sufficient conditions computed in Section 7 so that the standard construction starting at the standard $(x_0,y_0)=(r_1-y_1)$ arrives at $(-R+\rho,0)$ in exactly $\frac{n+1}{2}$ steps but not necessarily just one time in $\frac{n+1}{2}$ steps. Now any standard construction that arrives at $(-R+\rho,0)$ in exactly $\frac{n+1}{2}$ steps must either pass through $(-R+\rho,0)$ just one time in exactly $\frac{n+1}{2}$ steps or it has already arrived at $(-R+\rho,0)$ in exactly $\frac{n_i+1}{2}$ steps and passed through $(-R+\rho,0)$ just one time in $\frac{n_i+1}{2}$ steps for some n_i in our list $3\leq n_1<n_2<\cdots<n_k<n$.

We now eliminate all r-traces of the polynomials $P_{n_1}=0, P_{n_2}=0,\cdots,P_{n_k}=0$ from the polynomial $R_n=0$ by using Algorithm 1 of Section 3 with emphasis on Comment 1 at the end of Algorithm 1. If we use Comment 1 a simple division may be all that we need to use Algorithm 1.

The polynomial divisor of $R_n(R,r,\rho)=0, R>r+|\rho|, r>0$, that remains after all r-traces of $P_{n_1}=0, P_{n_2}=0,\cdots,P_{n_k}=0$ have been removed from $R_n(R,r,a)=0$ will be the required polynomial $P_n(R,r,\rho)=0, R>r+|\rho|, r>0$, that solves Problem 1. The solution to Problem $1'$ is almost exactly the same. ∎

Solution to Problem 2 Suppose $n \geq 4$, n even, is fixed and suppose Problem 2 has been solved for all $\overline{n}$ where $4 \leq \overline{n} < n$ and $\overline{n}$ is even.

We wish to calculate $P_n(R, r, \rho) = 0, R > r + |\rho|, r > 0$. The solution is almost exactly the same as Problems $1, 1'$. Suppose $4 \leq n_1 < n_2 < \cdots < n_k < n$ is the list of all positive even integers $\overline{n}$ that lie in $4 \leq \overline{n} < n$ with the following property $(**)$.

Of course, as always for each n_i in the list, $P_{n_i}(R, r, \rho) = 0, R >$ r$+|\rho|, r > 0$, are the necessary and sufficient conditions so that the standard construction starting at the standard $(x_0, y_0) = (r, -y_1)$ arrives at $\left(-r, +\sqrt{R^2 - (-r-\rho)^2}\right)$ in exactly $\frac{n_i}{2}$ steps and passes through $\left(-r, \sqrt{R^2 - (-r-a)^2}\right)$ just one time in $\frac{n_i}{2}$ steps.

Property $(**)$ For each n_i in the list, we require these (Poncelet) n_i-gons or n_i-stars to also arrive at $\left(-r, \sqrt{R^2 - (-r-\rho)^2}\right)$ in exactly $\frac{n}{2}$ steps.

In this note, for each even $4 \leq n$, we compute the above list $4 \leq n_1 < n_2 \cdots < n_k < n$ of even n_i's ad hoc by simply checking each even $4 \leq \overline{n} < n$ to see if $\overline{n}$ has property $(**)$.

Now $R_n(R, r, \rho) = 0, R > r + |\rho|, r > 0$, are the necessary and sufficient conditions computed in Section 7 so that the standard construction starting at the standard $(x_0, y_0) = (r, -y_1)$ arrives at $\left(-r, +\sqrt{R^2 - (-r-\rho)^2}\right)$ in exactly $\frac{n}{2}$ steps but not necessarily just one time in $\frac{n}{2}$ steps.

Now any standard construction that arrives at $\left(-r, +\sqrt{R^2 - (-r-\rho)^2}\right)$ in exactly $\frac{n}{2}$ steps must arrive at $\left(-r, +\sqrt{R^2 - (-r-\rho)^2}\right)$ just one time in $\frac{n}{2}$ steps or it has already arrived at $\left(-r, +\sqrt{R^2 - (-r-\rho)^2}\right)$ in exactly $\frac{n_i}{2}$ steps and passed through $\left(-r, +\sqrt{R^2 - (-r-\rho)^2}\right)$ just one time in $\frac{n_i}{2}$ steps for some n_i in our list $4 \leq n_1 < \cdots < n_k < n$.

As in Problems $1, 1'$, we now eliminate all r-traces of the polynomial $P_{n_1} = 0, P_{n_2} = 0, \cdots, P_{n_k} = 0$ from $R_n(R, r, \rho) = 0$ using Algorithm 1 of Section 3 with emphasis on

Comment 1 at the end of Algorithm 1. If we use Comment 1, a simple division may be all that we need to use Algorithm 1.

The polynomial divisor of $R_n(R, r, \rho) = 0, R > r + |\rho|, r > 0$, that remains after all r-traces of $P_{n_1}, P_{n_2}, \cdots, P_{n_k}$ have been removed from $R_n(R, \rho, P), R > r + |\rho|, r > 0$ will be the required polynomial $P_n(R, r, \rho) = 0, R > r + |\rho|, r > 0$ that solves problem 2. ■

Solution to Main Problem 1 As stated in Section 6, the solutions to Problems $1, 1', 2$ give the solution $P_n^* = P_n, P_n^* = \overline{P}_n, P_n^* = P_n$ where $P_n^*(R, r, \rho), R > r + |\rho|, r > 0$, is the polynomial solution to Main Problem 1. ■

9 Some Hand Calculated Examples

We solve Main Problem 1 for $n = 3, 4$ by hand.

Example 1($n = 3$) For $n = 3$, it is easy to see that $P_3^*(R, r, \rho) = P_3(R, r, \rho) = R_3(R, r, \rho) = 0$ where $R_3(R, r, \rho) = 0$ is the polynomial computed for Problem 1^* in Section 7 using two different methods. We now give both the long first method and the very short second method.

From Section 7, we see that the Problem 1^* equation $x_{\frac{n+1}{2}} = -R+\rho$ becomes $x_2 = -R+\rho$. The Problem 1^{**} equation $x_2 = R + \rho$ is degenerate.

From Section 5, $(x_1, Y_1, M_1) = (r, 1, 0)$ and we recall that $y_1^2 = R^2 - (r-\rho)^2$ and $M_2 = \frac{2x_1Y_1}{Y_1^2 \cdot y_1^2 - r^2} - M_1 = \frac{2r}{y_1^2 - r^2} = \frac{2r}{R^2 - 2r^2 + 2\rho r - \rho^2}$.

Using the recursion for x_2 of Section 5 and simplifying we see that $x_2 = -R + \rho$ becomes the following.

$$\begin{aligned}
&\frac{4r^2(2\rho - r)y_1^2 - 4ry_1^2(R^2 - 2r^2 + 2\rho r - \rho^2) + r(R^2 - 2r^2 + 2\rho r - \rho^2)^2}{4r^2y_1^2 + (R^2 - 2r^2 + 2\rho r - \rho^2)^2} \\
= \quad & -R + \rho.
\end{aligned}$$

This is equivalent to the following.

$$\begin{aligned}
&4ry_1^2\left[2\rho r - r^2 - R^2 + 2r^2 - 2\rho r + \rho^2\right] + r\left[R^2 - 2r^2 + 2\rho r - \rho^2\right]^2 \\
= \quad & 4ry_1^2(-Rr + \rho r) + \left[R^2 - 2r^2 + 2\rho r - \rho^2\right]^2(-R + \rho)
\end{aligned}$$

which is equivalent to $(*)$.

$(*)$

$$4ry_1^2\left(-R^2 + r^2 + Rr - \rho r + \rho^2\right) = -\left[R^2 - 2r^2 + 2\rho r - \rho^2\right]^2(R + r - \rho).$$

Since $y_1^2 = (R + r - \rho)(R - r + \rho)$ and $R + r - \rho = 0$ is an extraneous equation that contradicts $R > r + |\rho|, r > 0$, we see that $(*)$ is equivalent to the following

$$4r(-R + r - \rho)\left[-R^2 + r^2 + Rr - \rho r + \rho^2\right] = \left[R^2 - 2r^2 + 2\rho r - \rho^2\right]^2.$$

When we multiply this out and then simplify this becomes $R^4 - 4rR^3 + 4r^2R^2 - 2\rho^2R^2 + 4\rho^2rR + \rho^4 = 0$ which is equivalent to $\left[(R^2 - \rho^2) - 2rR\right]^2 = 0$. This is equivalent to $P_3^* = R^2 - \rho^2 - 2rR = 0$ which is the standard Euler's equation. The cononical form of Comment 1 would automatically catch this multiplicity. We now solve Example 1 by computing $P_3^* = P_3 = R_3^{'} = R_3$ by using the short second method of Problem 1^* of Section 7.

From $x_1 = r, Y_1 = 1, y_1^2 = (R - r + \rho)(R + r - \rho)$ and $M_2 = \frac{2r}{y_1^2 - r^2} = \frac{2r}{R^2 - 2r^2 + 2\rho r - \rho^2}$ we see that $x_{\frac{n-1}{2}} + R - \rho = Y_{\frac{n-1}{2}} M_{\frac{n+1}{2}} y_1^2$ becomes $x_1 + R - \rho = Y_1 M_2 y_1^2$, which is $r + R - \rho = \frac{2r \cdot (R - r + \rho)(R + r - \rho)}{R^2 - 2r^2 + 2\rho r - \rho^2}$.

Dividing out the extraneous equation $r+R-\rho=0$ this becomes $R^2-2r^2+2\rho r-\rho^2=2Rr-2r^2+2\rho r$ and we see that $P_3^*=P_3=R_3'=R_3=R^2-\rho^2-2rR=0$. ■

Note 2 We see that the second method is very superior to the first method. If we try to compute $R_3(R,r,\rho)=0$ for $n=3$ by the second method of Section 7, we see that $x_{\frac{n-1}{2}}-R-\rho=Y_{\frac{n-1}{2}}M_{\frac{n+1}{2}}y_1^2$ becomes $x_1-R-\rho=Y_1M_2y_1^2$ which is $r-R-\rho=\frac{2r}{y_1^2-r^2}y_1^2$. This is equivalent to $-(y_1^2-r^2)=2r(R+r-\rho)$ which simplifies to $\rho^2=R^2+2rR$. This equation is degenerate since we require $R>r+|\rho|, r>0$. However, we need to keep this equation $\rho^2-R^2-2Rr=0$ since this factor will divide out of some of the higher level equations that we will encounter. In particular, we use $\rho^2=R^2+2rR$ when we deal with $n=5$.

Example 2 $(n=4)$ We compute $P_4^*(R,r,\rho)=P_4=0$ for $n=4$. For $n=4$ it is easy to see that $P_4(R,r,\rho)=R_4'(R,r,\rho)=R_4(R,r,\rho)$ where $R_4'(R,r,\rho)$ is the polynomial computed in Problem 2* of Section 7. We note that $R_4(R,r,\rho)=R_4'(R,r,\rho)$ since $R_{n-2}=R_2=2r$ is degenerate and we are going to divide r out of R_4' anyway. Using the formula for x_2 given in Example 1, we see that $x_2=-r$ becomes the following.

$\frac{4r^2y_1^2(2\rho-r)-4ry_1^2\left(R^2-2r^2+2\rho r-\rho^2\right)+r\left(R^2-2r^2+2\rho r-\rho^2\right)^2}{4r^2y_1^2+(R^2-2r^2+2\rho r-\rho^2)^2}=-r$ which is equivalent to the following.

$$4r^2y_1^2(2\rho-r)-4ry_1^2(R^2-2r^2+2\rho r-\rho^2)+r(R^2-2r^2+2\rho r-\rho^2)^2=$$
$$-r\left[4r^2y_1^2+(R^2-2r^2+2\rho r-\rho^2)^2\right].$$

Dividing out the extraneous $r=0$, this becomes $4r(2\rho-r)y_1^2-4y_1^2(R^2-2r^2+2\rho r-\rho^2)+4r^2y_1^2=-2[R^2-2r^2+2\rho r-\rho^2]^2$.

Dividing out 2 and simplifying we have $2y_1^2(R^2-2r^2-\rho^2)=(R^2-2r^2+2\rho r-\rho^2)^2$.

Using $y_1^2=R^2-r^2+2\rho r-\rho^2$ this becomes $2(R^2-r^2+2\rho r-\rho^2)(R^2-2r^2-\rho^2)=(R^2-2r^2+2\rho r-\rho^2)^2$.

Multiplying out and simplifying we have $R^4 - 2\rho^2 R^2 + \rho^4 = 2\rho^2 r^2 + 2r^2 R^2$ which is $\left(R^2 - \rho^2\right)^2 = 2r^2\left(R^2 + \rho^2\right)$. This is the standard quadrilateral formula. ■

10 Some Computer Generated Examples

We solve Main Problem 1 for $n = 5, 6, 7$ by using a computer.

Example 3 ($n = 5$) We first compute $R_5'(R, r, \rho) = 0$ of Problem 1^* by using the equality $(*)\, x_2 + R - \rho - Y_2 M_3 y_1^2 = 0$ which is equivalent to $(*)\, (x_2 - r) + (R + r - \rho) - Y_2 M_3 y_1^2 = 0$. Dividing $R + r - \rho$ out of $(*)$ and using a computer, we arrive at $R_5'(R, r, \rho) = 16\rho^2 R^2 r^4 + 8R\left(R^2 - \rho^2\right)^2 r^3 - 8R^2\left(R^2 - \rho^2\right)^2 r^2 + \left(R^2 - \rho^2\right)^4 = 0$. From Problem 1^* of Section 7, we know that $R_3(R, r, \rho) = 2Rr + \rho^2 - R^2 = 0$ will also solve $(*)$ where $R_3 = 0$ was computed in Example 1 ($n = 3$). Therefore, we must eliminate all r-traces of $R_3 = 0$ from $R_5' = 0$. We can do this by dividing R_5' by R_3 and letting $R_5(R, r, \rho)$ be the quotient. This gives $R_5(R, r, a) = 8\rho^2 R r^3 + 4R^2\left(R^2 - \rho^2\right) r^2 - 2R\left(R^2 - \rho^2\right)^2 r - \left(R^2 - \rho^2\right)^3 = 0$. It is easy to show that $R_5 = 0$ is irreducible in the rational field. We now let $R = 1$ and by symmetry suppose $0 \leq \rho < 1$. We know by Descartes' law of signs that $R_5 = 0$ has one positive r-root for each fixed $0 \leq \rho < 1$. For each fixed $0 \leq \rho < 1$, we show that $R_5 = 0$ has one r-root that satisfies $0 < r < 1 - \rho$. Now $R_5(R, r, \rho) = R_5(1, 0, \rho) = -(1 - \rho)^3 < 0$. We now show that $R_5(R, r, \rho) = R_5(1, 1 - \rho, \rho) > 0$. This is true if and only if $\left[8\rho^2 + 4(1 + \rho) - 2(1 + \rho)^2 - (1 + \rho)^3\right](1 - \rho)^3 > 0$. This is true if and only if $\left(1 - 3\rho + 3\rho^2 - \rho^3\right)(1 - \rho)^3 = (1 - \rho)\, 6 > 0$, which is true. From this we see that for each $0 \leq \rho < 1, R_5 = 0$ has one r-root that satisfies $0 < r < 1 - \rho$. Therefore, in general for each $R > |\rho| \geq 0$, we see that $R_5(R, r, \rho) = 0$ has one r-root that satisfies $R > r + |\rho|, r > 0$.

We let $P_5^* = R_5$ where $P_5^* = 0$ is one solution to Main Problem 1. We next compute $\overline{R}_5'(R, r, \rho) = 0$ by using the equality $(**)\, x_2 - R - \rho - Y_2 M_3 y_1^2 = 0$ which is equivalent to $(**)\, (x_2 - r) - R + r - \rho - Y_2 M_3 y_1^2 = 0$. Dividing $r - R - \rho$ out of $(**)$ and using a computer we arrive at $\overline{R}_5'(R, r, \rho) = 16\rho^2 R^2 r^4 - 8R\left(R^2 - \rho^2\right)^2 r^3 - 8R^2\left(R^2 - \rho^2\right)^2 r^2 + \left(R^2 - \rho^2\right)^4 = 0$.

Now $\overline{R}_3(R, r, \rho) = -2Rr + \rho^2 - R^2 = 0$ will also solve $(**)$. Therefore, we must eliminate all r-traces of $\overline{R}_3 = 0$ from $\overline{R}_5' = 0$. We can do this by dividing $\overline{R}_5'$ by $\overline{R}_3$ and letting $\overline{R}_5(R, r, \rho)$ be the quotient. This gives $\overline{R}_5(R, r, \rho) = -8\rho^2 R r^3 + 4R^2\left(R^2 - \rho^2\right) r^2 + 2R\left(R^2 - \rho^2\right)^2 r - \left(R^2 - \rho^2\right)^3 = 0$.

It is easy to show that $\overline{R}_5 = 0$ is irreducible in the rational field. We now let $R = 1$ and by symmetry suppose $0 < \rho < 1$. We know by Descartes' law of signs tht $\overline{R}_5 = 0$ has zero or two positive r-roots for each fixed $0 < \rho < 1$. For each fixed $0 < \rho < 1$ we show that $\overline{R}_5 = 0$ has one r-root that satisfies $0 < r < 1 - \rho$. ($\rho = 0$ is easy to deal with). Now $\overline{R}_5(R, r, \rho) = \overline{R}_5(1, +\infty, \rho) < 0$. Also, $\overline{R}_5(R, r, \rho) = \overline{R}_5(1, 0, \rho) < 0$. If we show that $\overline{R}_5(R, r, \rho) = \overline{R}_5(1, 1 - \rho, \rho) > 0$, then it will follow that for each $0 < \rho < 1, \overline{R}_5 = 0$ has one r-root that satisfies $0 < r < 1 - \rho$. Now $\overline{R}_5(R, r, \rho) = \overline{R}_5(1, 1 - \rho, \rho) > 0$ if and only if $\left[-8\rho^2 + 4(1 + \rho) + 2(1 + \rho)^2 - (1 + \rho)^3\right](1 - \rho)^3 > 0$. This is true if and only if $\left[4\left(1 + \rho - 2\rho^2\right) + (1 + \rho)^2(2 - (1 + \rho))\right](1 - \rho)^3 = \left[4(1 + 2\rho)(1 - \rho) + (1 + \rho)^2(1 - \rho)\right](1 - \rho)^3 > 0$ which is true.

Therefore, in general for each $R > |\rho| \geq 0$, we see that $\overline{R}_5(R, r, \rho) = 0$ has one r-root that satisfies $R > r + |\rho|, r > 0$. We let $P_5^* = \overline{R}_5$ where $P_5^* = 0$ is that second solution to Main Problem 1.■

Therefore, $P_5^* = R_5$ and $P_5^* = \overline{R}_5$ are the two solutions to Main Problem 1 for $n = 5$.

Example 4 ($n = 6$) From Problem 2^*, we define the equation $x_{\frac{n}{2}} = x_3 = -r$. We do this with a computer. From Comment 2 we divide r out of $x_3 = -r$ where $r = 0$ is an extraneous factor since it contradicts $r > 0$. This defines a polynomial equation $R_6'(R, r, \rho) = 0$ which we store in the computer. From Problem 2^*, to compute $R_6 = 0$ we must eliminate all r-traces of $R_4(R.r.\rho) = 2(R^2 + \rho^2) r^2 - (R^2 - \rho^2)^2 = 0$ from $R_6'(R, r, \rho) = 0$. This can be done by a single division $\frac{R_6'(R,r,\rho)}{R_4(R,r,\rho)} = R_6(R, r, \rho)$.

That is, $R_6'(R, r, \rho) = R_4(R, r, \rho) \cdot R_6(R, r, \rho) = \left[(R^2 - \rho^2)^2 - 2(\rho^2 + R^2) r^2\right] \cdot R_6(R, r, \rho)$ where $R_6(R, r, \rho) = 16\rho^2 R^2 r^4 + 4(\rho^2 + R^2)(R^2 - \rho^2)^2 r^2 - 3(R^2 - \rho^2)^4 = 0$. We now let $R = 1, 0 \leq \rho < 1$. We know from Descartes' Law of signs that $R_6(R, r, \rho) = R_6(1, r, \rho) = 0$ has one positive r-root for each fixed $0 \leq \rho < 1$. We now show that for each fixed $0 \leq \rho < 1, R_6 = 0$ has one r-root that satisfies $0 < r < 1 - \rho$. Now $R_6(R, r, \rho) = R_6(1, 0, \rho) = -3(1 - \rho^2)^4 < 0$. We now show that $R_6(R, r, \rho) = R_6(1, 1 - \rho, \rho) > 0$ which will finish the proof. Now $R_6(1, 1 - \rho, \rho) > 0$ is true if and only if $16\rho^2(1 - \rho)^4 +$ $4(1 + \rho^2)(1 + \rho)^2(1 - \rho)^4 - 3(1 + \rho)^4(1 - \rho)^4 > 0$. This is equivalent to $\left[16\rho^2 + 4(1 + \rho^2)(1 + \rho)^2 - 3(1 +\right.$ $[\rho^4 - 4\rho^3 + 6\rho^2 - 4\rho + 1](1 - \rho)^4 = (1 - \rho)^8 > 0$ which is true. From Problem 2 of Section 8, we know that $P_6^* = P_6 = R_6(R, r, a)$ where $P_6^* = 0$ solves Main Problem 1 for $n = 6$. ■

<u>Example 5 $(n = 7)$</u> We first compute $R_7'(R, r, \rho) = 0$ of Problem 1^* by using the equality $(*), x_3 + R - \rho - Y_3 M_4 y_1^2 = 0$ which is equivalent to $(*)\,(x_3 - r) + (R + r - \rho) - Y_3 M_4 y_1^2 = 0$.

Dividing $R + r - \rho$ out of $(*)$, we arrive at $R_7'(R, r, \rho) = 0$ and we store this in the computer. From Problem 1^* of Section 7, we know that $R_5(R, r, \rho) = 8\rho^2 R^2 r^3 + 4R^2(R^2 - \rho^2)^2 r^2 - 2R(R^2 - \rho^2)^2 r - (R^2 - \rho^2)^3 = 0$ will also solve $(*)$ where $R_5 = 0$ was computed in Example 3 $(n = 5)$. Therefore, we must eliminate all r-traces of $R_5 = 0$ from $R_7' = 0$. We can do this by dividing R_7' by R_5 and letting $R_7(R, r, \rho)$ be the quotient. This gives $R_7(R, r, \rho) = 64\rho^2 R^4 r^6 - 32\rho^2 R(R^2 - \rho^2)(R^2 + \rho^2) r^5 - 16\rho^2 R^2(R^2 - \rho^2)^2 r^4 +$

$8R\left(R^2+3\rho^2\right)\left(R^2-\rho^2\right)^3 r^3 - 4R^2\left(R^2-\rho^2\right)^4 r^2 - 4R\left(R^2-\rho^2\right)^5 r + \left(R^2-\rho^2\right)^6 = 0.$

The equation $\overline{R}_7\left(R,r,\rho\right)=0$ is the same as $R_7\left(R,r,\rho\right)=0$ except that $\overline{R}_7\left(R,r,\rho\right) = R_7\left(-R,r,\rho\right) = R_7\left(R,-r,\rho\right) = 0$. The solution to Main Problem 1 is $P_7^* = R_7$ and $P_7^* = \overline{R}_7$. If we let $R=1, 0<\rho<1$, we can use a computer to show that $R_7\left(R,r,\rho\right) = R_7\left(1,r,\rho\right) = 0$ has two real r-roots r_1, r_2 that satisfy $0<r_1<r_2<1-\rho$. These two r-roots give the 7-gon and a 7-star that goes around $(0,0)$ three times when $R=1$. The equation $\overline{R_7}(R,r,\rho) = \overline{R}_7\left(1,r,\rho\right)$ has one r-root r_3 that satisfies $0<r_3<1-\rho$. This r_3 gives a 7-star that goes around $(0,0)$ two times when $R=1$. ■

As is consistent with the general pattern, $R_7\left(R,r,\rho\right) = R_7\left(1,1-\rho,\rho\right) = \left(1-\rho\right)^{12}$. We recall that $R_6\left(1,1-\rho,\rho\right) = \left(1-\rho\right)^8$. Also, $R_5\left(1,1-\rho,\rho\right) = \left(1-\rho\right)^6$.

Also, $R_4\left(1,1-\rho,\rho\right) = 2\left(1-\rho\right)^2\left(1+\rho^2\right) - \left(1-\rho^2\right)^2 = \left(1-\rho\right)^4$.

Also, $R_3\left(1,1-\rho,\rho\right) = 2\left(1-\rho\right) - \left(1-\rho^2\right) = \left(1-\rho\right)^2$.

In general $R_i\left(1,1-\rho,\rho\right) = \left(1-\rho\right)^{2n_i}$ where n_i is the r-degree of $R_i\left(R,r,\rho\right)$.

11 Discussion

It is fairly obvious that the polynomials R_3, R_4, R_5, R_6, R_7 that we have computed are members of a family. For example they all have powers of 2, i.e. $2, 4, 8, 16, 32, 64, \cdots$ appearing in them. They also have $\left(R^2-\rho^2\right), \left(R^2-\rho^2\right)^2, \ldots$ appearing in them. In general, they just look alike in some ways. However, the only true invariant that we have discovered for R_3, R_4, R_5, R_6, R_7 is that each R_i satisfies $R_i\left(R,r,\rho\right) = R_i\left(1,1-\rho,\rho\right) = \left(1-\rho\right)^{2n_i}$, where n_i is the $r-$ degree of $R_i\left(R,r,\rho\right)$.

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