Nim with Twists

Arthur Holshouser
3600 Bullard St.
Charlotte, NC,
USA
Harold Reiter
Department of Mathematics,
University of North Carolina Charlotte,
Charlotte, NC 28223, USA
hbreiter@email.uncc.edu

1 Abstract

In November 2001, Collage Math Journal, Problem 714, we posed as a problem an n-pile Nim game in which on each turn the opposing player is allowed to block some of the moving player's moves. The basic idea of the solution generalizes easily and naturally. In the present paper we will explain in intuitive language how we generalized this basic idea as far as possible. Our goal here is to provide a framework which incorporates many of the new features we have been studying, like blocking [5], [6], and tampering. These are examples of games with an additional feature called a *Muller twist*. We say that we put a Muller twist on a familiar game if we modify that game so that each move in the familiar game is followed by a constraint on the next player's move. The name honors Blaise Muller, the inventor of the game Quarto, published by Gigamic, which was one of the five Mensa Games of the Year in 1993. See [3] and [7].

2 An Intuitive Generalization

. Let N denote the natural numbers $\{0, 1, 2, 3, \cdots\}$. Each $x \in N$ is called a vertex. At the beginning of the game, a single counter is placed on some vertex. Two players Art and Beth alternate moving the counter from vertex to vertex. The winner is the player who makes the last move.

Suppose the counter is on x, and suppose it is Art's turn to move. The rules for Art's move depend on x, and these rules can vary as x varies over N. This means that no matter how the counter arrived at x, the rules have no memory of the past. A move for Art is a sequence of 'steps' which we call an *encounter*. The remarkable feature of the game is the generality of the composition of an encounter. If Art, using perfect play, cannot force the counter to end up on a smaller vertex at the end of the encounter, we say that x is a terminal vertex. The rules for Art's move specify some type of step sequence involving Art and Beth that (for the present) does not involve chance. Such an encounter, starting at vertex x for Art can go something like the following.

1. Beth blocks two of Art's options before Art moves the counter.
2. Then Art moves the counter to any smaller vertex that is at least $\lfloor x/2 \rfloor$ and that Beth has not blocked.
3. Then Art blocks one of Beth's options before Beth moves the counter.
4. Then Beth moves the counter to a smaller vertex by subtracting $1, 2, 3$ or 4 that Art has not blocked.

5. Art moves the counter to a smaller vertex by subtracting 2 or 5.
6. If the sum of the subtractions in the steps 2, 4 and 5 is less than 5, then Art has the option of adding 4.
7. Beth has the option of moving or not moving the counter. If she moves, she must add 1 or 7.
8. Beth blocks one of Art's options before Art moves the counter.
9. Then Art moves the counter by subtracting from 1 up to 10 provided Beth has not blocked that option.
10. Beth has the option of subtracting the same number that Art subtracted in step 5.

Note the entire sequence of steps (the encounter) is regarded as a single move for Art when the counter starts at x.

3 The Theory

If some steps of the encounter cannot be carried out, the counter remains fixed for that step. When Art's encounter is over, we consider Art's move as having taken place if the counter has moved to a smaller vertex at the end of the encounter. Otherwise, we consider Art's move as not having taken place and player Beth wins the game.

Of course, an encounter for Art might have a trillion steps for some x's. The rules for Beth's move (starting at x) consist of interchanging Art and Beth in the rules. That is, the rules are symmetric or impartial.

If x is a terminal vertex, the Nim value of x is $g(x) = 0$. Of course, $g(0) = 0$. Suppose x is a non-terminal vertex, and (by symmetry) suppose it is Art's turn to move and suppose $g(0), g(1), g(2), \cdots, g(x-1)$ have been computed. The Nim value of x, called $g(x)$, is computed as follows.

First, we define the set S_x as follows:

$$S_x = \{n \in N| \text{ Art using perfect play can force the counter, starting at } x \text{ to end in a}$$
$$\text{vertex } y \text{ such that } 0 \leq y < x \text{ and } g(y) = n\}.$$

Define $g(x) = mexS_x$ where $mexS_x$ is the smallest member of $N \backslash S_x$. For example, $mex\{0, 1, 3, 7\} = 2$.

We need the following axiom.

<u>Axiom</u>. If Art's encounter starts at x, then for each positive integer n, either

1. Using perfect play, Art can force the counter to end up inside the set $\{y|\ 0 \leq y \leq x-1,\ \text{and}\ g(y) = n\}$ or

2. using perfect play, Beth can force the counter to end up outside the set $\{y|\ 0 \leq y \leq x-1,\ \text{and}\ g(y) = n\}$.

All the examples below satisfy the axiom, and we believe that all encounters not involving chance satisfy the axiom. Also, we point out that an encounter may involve chance as long as the axiom holds. Example 5 below shows how chance moves might be incorporated. Note that a single move for Art can be viewed as a mini game in which Art wishes to end up on a particular Nim value and Beth wishes to prevent Art from ending up on a particular Nim value.

The Nim values of these complicated games can now be used in the standard way to play composite games. In a composite game, we have n of these games, and a counter lies on a vertex in each of the games. Two players alternate moving, and the winner makes the last move. Each player in his turn first chooses one of the games, and then makes a move in that game. If the counters lie on vertices $x_1, x_2, \cdots, x_n$ in the different games, a position can be denoted $(x_1, x_2, \cdots, x_n)$. The Nim value of $(x_1, x_2, \cdots, x_n)$ is computed by $g(x_1, x_2, \cdots, x_n) = g(x_1) \oplus g(x_2) \oplus \cdots \oplus g(x_n)$ where $\oplus$ is the usual nim sum. A position $(x_1, x_2, \cdots, x_n)$ is balanced if $g(x_1, x_2, \cdots, x_n) = g(x_1) \oplus g(x_2) \oplus \cdots \oplus g(x_n) = 0$. The balanced positions are winning positions for the second player and the proof and play are virtually identical to standard n-pile Nim. The Nim values are used analogously to the pile sizes in standard n-pile Nim. The reader is invited to read the theory of standard Nim in [2] and [4].

4 Examples

Example 1 (Tampering Nim). This is a composite game, and we start with n piles of counters. Each pile is a game. Art and Beth alternate moving, and the winner makes the last move. Suppose it is Art's turn to move. The first thing Art does is to choose one of the non-empty piles. An encounter for Art consists of the following two steps.

1. Art removes any number of counters from the chosen pile that he wishes.

2. Beth has the option of removing one move counter from the chosen pile if any counters remain in the chosen pile.

An encounter for Beth is defined the same way.

Example 2 (Blocking Nim). We again have n piles of counters. Art and Beth alternate, and the winner makes the last move. The first thing Art does is choose one of the non-empty piles having 2 or more counters. However, before Art moves, Beth must block one of Art's moves. For example, if the chosen pile has 5 counters, Beth might prevent Art from removing 3 counters. This would leave Art the option of removing 1,2,4,5 counters. An encounter for Beth is the same.

Example 3 (Reverse Nim). We again have n piles of counters. Art and Beth alternate, and the winner makes the last move. Suppose it is Art's move. The first thing Art does is to choose one of the non-empty piles. A encounter for Art then consists of the single step of letting Beth remove any number that Beth wishes from the chosen pile. This means that on Art's move, Art picks the pile and Beth does the removing of counters. Of course, if Beth removes the last counter on the table, then Art wins. An encounter for Beth is defined the same way.

Example 4 (Blocking and Tampering Nim). We again have n piles of counters. Art and Beth alternate and the winner makes the last move. Suppose it is Art's move. An encounter for Art consists of the following sequence.

1. Art chooses one of the piles with at least two counters.
2. Beth must block one of Art's moves from this pile.
3. Art removes any number of counters except the blocked number.
4. Beth has the option of removing one more counter.

Example 5 (Nim with dice). Again its n pile Nim with each alternating player required to remove an odd number of counters as determined by the roll of a die.

Example 6 (Doubling Nim). Again there are n piles. Suppose its Art's turn. Art chooses any pile and removes any number of counters from this pile. If Art removes k counters, Beth has the option of tampering with Art's move by removing an additional k counters from the same pile if that pile has at least k counters left.

References

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