Misère Games

Arthur Holshouser
3600 Bullard St.
Charlotte, NC 28208, USA

Harold Reiter
Department of Mathematics,
University of North Carolina Charlotte,
Charlotte, NC 28223, USA
`hbreiter@email.uncc.edu`

Abstract

The theory of last-player winning counter pickup games is well-known. See [1] and [2]. The corresponding misère games in which the last player loses are less well understood. In this note, we define a special class of combinatorial games and find the winning strategies for all composite games with these special games as components. In the first section we recall the method of using Nim values of component games to solve a composite game. In the second section, we define *special* games and find winning strategies for misère games.

Section 1

Definition. A finite impartial game G played under normal rules of play is called a *regular* game. This means that

1. Two players alternate moving,
2. There are no infinite sequence of moves,
3. Both players have the same moves available, and
4. The winner is the last player to make a move.

Such a game can be thought of as a directed acyclic graph. Each *vertex* of the graph corresponds to a position in the game and each *directed edge* corresponds to a move. The *followers* of a vertex are those positions joined to it by an outgoing edge. We will briefly say that G is a regular impartial game.

Nim Values.

The *mex* (minimum excluded value) of a finite set of nonnegative integers is the least nonnegative integer not in the set. For example, $mex\,\{1,2,4,0\} = 3, mex\,\{2,4,5\} = 0, mex\,\{\quad\} = 0$. The Nim value of a position, denoted by $g(n)$, is the mex of the Nim values of its followers. A position with no followers (a terminal position) has Nim value 0. It is easy to see that the winning strategy is to move to a position with Nim value 0, for then the opponent either has no move at all and loses immediately, or must move to a position with Nim value greater than 0 and so must eventually lose.

Composite Games. Composite games, denoted $G = G_1 \oplus G_2 \oplus \cdots \oplus G_k$, are games that have several components. Two players alternate moving. Each player on his turn selects a component game G_i in which a legal move can be made and makes a legal move in that game. The winner is the last player to move. The Nim values of the composite game is the Nim sum $\oplus$ of the Nim values of the component games. The Nim sum is obtained by writing the integers in binary and adding modulo 2 without carrying. For example $6 \oplus 3 = 110_2 \oplus 11_2 = 101_2 \equiv 5$ since $1 \equiv 1 (\text{mod } 2)$, $1 + 1 = 0 (\text{mod } 2)$, and $0 + 1 \equiv 1 (\text{mod } 2)$.
Strategy. The balanced positions are those positions whose Nim values are 0. The unbalanced positions are those positions whose Nim values are not zero.

If a position is balanced, it will always become unbalanced after the moving player moves. This follows from the definition of *mex* since if a player moves from n_i to m_i in G_i, then $g(n_i) \neq g(m_i)$.

Also, if a position is unbalanced, the moving player can always move to a balanced position. Such a winning move can always be selected from the component G_i that contributes the left-most 1 in the Nim sum of the component values. This follows from the definition of *mex* since if $g(n_i) \geq 1$ in game G_i, the moving player can move in G_i to a vertex m_i having any of the values $\{0, 1, 2, \cdots, g(n_i) - 1\}$. In particular, the moving player can move to a position m_i whose value is the sum of the Nim values of the other components. Of course, all terminal positions have a Nim value of $0 \oplus 0 \oplus \cdots \oplus 0 = 0$, which is balanced.

Section 2

Misère version of a Game. The misère version of a regular impartial game G_i is played by the same rules as G_i except the loser is the player who makes the last move.

The misère version of a composite game $G_1 \oplus G_2 \oplus \cdots \oplus G_k$ is played by the same rules as $G_1 \oplus G_2 \oplus \cdots \oplus G_k$ except the loser is the last player to move.
Special Games. Suppose G is a regular impartial game. We say that G_i is *special* if for each position n in G if $g(n) = 0$ then (1) n is a terminal position or (2) there exists a follower m of n such that $g(m) = 1$.

Problem 1 . *Suppose $G_1, G_2, \cdots, G_k$ are special, regular impartial games. Find a strategy for playing the misère version of $G_1 \oplus G_2 \oplus \cdots \oplus G_k$.*

Solution 1 *Let $(n_1, n_2, \cdots, n_k)$ denote an arbitrary position in $G_1 \oplus G_2 \oplus \cdots \oplus G_k$. We will first define the balanced positions.*

A. *If each $g(n_i) \in \{0, 1\}$, then $(n_1, n_2, \cdots, n_k)$ is balanced if and only if $g(n_1) \oplus g(n_2) \oplus \cdots \oplus g(n_k) = 1$.*

B. *If at least one $g(n_i) \notin \{0, 1\}$, then $(n_1, n_2, \cdots, n_k)$ is balanced if and only if $g(n_1) \oplus g(n_2) \oplus \cdots \oplus g(n_k) = 0$.*

Let B, U denote the balanced and unbalanced positions respectively.

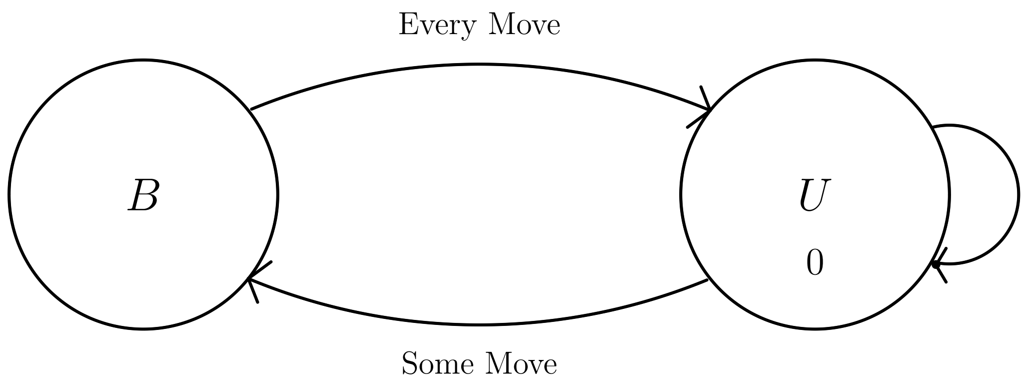

Fig. 1

We note that all terminal positions, which we denote 0, are unbalanced. We will prove the following which we have illustrated in Fig. 1.

1. If $(n_1, n_2, \cdots, n_k)$ is balanced, then all moves must be to an unbalanced position.

2. If $(n_1, n_2, \cdots, n_k)$ is unbalanced, and non-terminal, then there exists a move to a balanced position.

From (1), (2) it follows that if $(n_1, n_2, \cdots, n_k)$ is the initial position in the game, then

(a) if $(n_1, n_2, \cdots, n_k)$ is balanced, the first moving player will lose if the opposing player plays perfectly.

(b) if $(n_1, n_2, \cdots, n_k)$ is unbalanced, then the first moving player will win with perfect play.

We now prove (1), (2).

Proof of (1). For the balanced position $(n_1, n_2, \cdots, n_k)$, we consider A and B.

A. First, we suppose each $g(n_i) \in \{0, 1\}$. Because the positions are balanced, $g(n_1) \oplus g(n_2) \oplus \cdots \oplus g(n_k) = 1$. By symmetry suppose the moving player moves in game G_1 which must be non-terminal, of course. If $g(n_1) = 0$, by the definition of *mex* the moving player must move to m_1 with $g(m_1) = 1$ or $g(m_1) \geq 2$.

   In either case the new position $(m_1, n_2, n_3, \cdots, n_k)$ is unbalanced.

   If $g(n_1) = 1$ by the definition of *mex* the moving player must move to m_1 with $g(m_1) = 0$ or $g(m) \geq 2$. In either case the new position $(m_1, n_2, n_3, \cdots, n_k)$ is unbalanced.

B. Next, suppose at least one $g(n_i) \notin \{0, 1\}$. Then $g(n_1) \oplus g(n_2) \oplus \cdots \oplus g(n_k) = 0$, which implies there must also be a $j \neq i$ such that $g(n_j) \notin \{0, 1\}$. Now after the moving player moves, there must still exist a game G_j such that $g(n_j) \notin \{0, 1\}$. By the definition of *mex*, after the moving player moves it will be impossible for $g(\bar{n}_1) \oplus g(\bar{n}_2) \oplus \cdots \oplus g(\bar{n}_k) = 0$ where $(\bar{n}_1, \bar{n}_2, \cdots, \bar{n}_k)$ is the new position. Therefore, $(\bar{n}_1, \bar{n}_2, \cdots, \bar{n}_k)$ is unbalanced.

Proof of (2). For the unbalanced non-terminal position $(n_1, n_2, \cdots, n_k)$, we consider the two condition A and B.

A. Suppose at least one $g(n_i) \notin \{0,1\}$.

Subcase a. Only one $g(n_i) \notin \{0,1\}$. Since $g(n_i) \geq 2$, by the definition of *mex* the moving player can move to an m_i such that $g(m_i) = 0$ and move to an $\overline{m_i}$ such that $g(\overline{m_i}) = 1$. This easily implies that he can move to a balanced position.

Subcase b. Two or more $g(n_i) \notin \{0,1\}$. By the definition of *mex*, the moving player (as in Bouton's Nim) moves to a position $(\overline{n_1}, \overline{n_2}, \ldots, \overline{n_k})$ such that $g(\overline{n_1}) \oplus g(\overline{n_2}) \oplus \cdots \oplus g(\overline{n_k}) = 0$ which is a balanced position.

B. Suppose all $g(n_i) \in \{0,1\}$. Since $(n_1, n_2, \cdots, n_k)$ is unbalanced, we have $g(n_1) \oplus g(n_2) \oplus \cdots \oplus g(n_k) = 0$. Now since $(n_1, n_2, \cdots, n_k)$ is non-terminal, let n_i be a non-terminal vertex in a game G_i. If $g(n_i) = 1$ by the definition of *mex* the moving player can move to a m_i with $g(m_i) = 0$ which balances the game. If $g(n_i) = 0$, by the definition of a special game, the moving player can move to m_i with $g(m_i) = 1$ which again balances the game. ■

References

[1] Berlekamp, Conway, and Guy, Winning Ways, Academic Press, New York, 1982.

[2] Richard K. Guy, Fair Game, 2nd ed., COMAP, New York, 1989.