Creating Semi-Magic, Magic and Extra Magic $n \times n$ Squares when n is Odd

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1 Abstract

We define a simple algorithm for creating large numbers of semi-magic $n \times n$ squares when n is odd. Special cases of the algorithm can also be used to easily create magic and extra magic (or panmagic) $n \times n$ squares when n is odd. The algorithm can also create extremely magic $n \times n$ squares when $n \geq 5$ is prime. These extremely magic $n \times n$ squares have $n!$ standard magic n-element subsets, and these $n!$ standard magic n-element subsets include the n rows and n columns as well as the $2n$ generalized diagonals.

2 Introduction

Suppose the positive integers $1, 2, 3, \cdots, n^2$ are assigned to the n^2 positions of a $n \times n$ matrix. Also, suppose the sums of the n entries in each row and in each column have the common value $\frac{1}{n}\left[1+2+3+\cdots+n^2\right] = \frac{n}{2}\left(n^2+1\right)$.

This arrangement of $1, 2, \cdots, n^2$ is called a semi-magic square. If the sums of the n entries in each of the two main diagonals also have this common value $\frac{n}{2}\left(n^2+1\right)$, the arrangement is called a magic square. Thus, the following give two semi-magic squares with the second also being a magic square.

1	5	9
8	3	4
6	7	2

4	3	8
9	5	1
2	7	6

Fig. 1

We define extra magic squares in Section 9.

In this note we define a simple algorithm for creating large numbers of semi-magic $n \times n$ squares when n is odd. Special cases of the algorithm can also be used to easily create magic and extra magic $n \times n$ squares when n is odd. In a subsequent paper we use the algorithm to create extrememly magic 5×5 squares.

3 Basic Matrix Notation

Suppose we have a $n \times n$ square matrix as in Fig. 1. We number the columns in the order $0, 1, 2, 3, \cdots, n-1$ with the left column numbered 0 and the right column numbered $n-1$.

Also, we number the rows in the order $0, 1, 2, \cdots, n-1$ with the top row numbered 0 and the bottom row numbered $n-1$. As in the usual matrix notation, let square (i, j) be the square in the i^{th} row from the top and the j^{th} column from the left. Thus, the top left square is denoted by $(0, 0)$, the top right square is denoted by $(0, n-1)$, the bottom left square is denoted by $(n-1, 0)$ and the bottom right square is denoted by $(n-1, n-1)$.

4 An Algorithm for Creating Semi-Magic $n \times n$ Squares when n is Odd

Let n be a fixed odd positive integer. Suppose the letters $a_0, a_1, a_2, \cdots, a_{n-1}$ are any arbitrary but fixed permutation of the positive integers $1, 2, 3, \cdots, n$. Also, the letters $A_0, A_1, A_2, \cdots, A_{n-1}$ are any arbitrary but fixed permutation of the integers $1, 2, 3, \cdots, n$.

First, we agree that $t, l \in \{1, 2, 3, \cdots, n-1\}$ and $(t, n) = 1, (l, n) = 1, (t+l, n) = 1$. That is, (t, n) are relatively prime, (l, n) are relatively prime and $(t+l, n)$ are relatively prime. Such integers t, l exist since n is odd. For example, we can let $t = l = 1$ where $t + l = 2$.

We now assign ordered pairs $\left(a_{x_i}, A_{y_j}\right)$ to the squares (i, j) of a $n \times n$ matrix where n is odd according to the following rules.

First, in the top row we assign in order the ordered pairs $(a_0, A_0), (a_1, A_1), (a_2, A_2), (a_3, A_3)$, newline $\cdots, (a_{n-1}, A_{n-1})$ starting with (a_0, A_0) in square $(0, 0)$. Thus, (a_0, A_0) is in square $(0, 0), (a_1, A_1)$ in in square $(0, 1), (a_2, A_2)$ is in square $(0, 2), \cdots, (a_{n-1}, A_{n-1})$ is in square $(0, n-1)$.

In each row $k, 1 \le k \le n-1$, we assign a_0 to the square $(k, (kl) \bmod n)$ where $(ke) \bmod n$ is the remainder when kl is divided by n. Of course, since (n, l) are relatively prime and $1 \le k \le n-1$, we see that $(ke) \bmod n \in \{1, 2, 3, \cdots, n-1\}$.

We now shorten this notation and simply say that in each row $k, 1 \le k \le n-1$, we assign a_0 to the square $(k.kl)$ where we agree that kl is computed mod n.

Also, in each row $k, 1 \le k \le n-1$, starting with the a_0 in square (k, kl) we assign $a_1, a_2, \cdots, a_{n-1}$ to the other squares in row k so that $a_0, a_1, a_2, \cdots, a_{n-1}$ remain in this order. Thus, if a_0 is assigned to square $(2, 4)$ in row 2 and if $n = 7$, we would have the following a_i's in row 2 starting with a_3 in square $(2, 0)$ and ending with a_2 in square $(2, 6)$: $a_3, a_4, a_5, a_6, a_0, a_1, a_2$. Of course, from this it is obvious that each row contains all of the lower case letters $a_0, a_1, \cdots, a_{n-1}$ exactly one time.

Now since e and n are relatively prime, we see that $\{kl : k \in \{1, 2, \cdots, n-1\}\} =$ $\{1, 2, 3, \cdots, n-1\}$. Also, since a_0 is assigned to square $(0, 0)$, we can now easily see that each of the n columns $0, 1, 2, 3, \cdots, n-1$ will contain a_0 exactly one time. Also, since each row contains $a_0, a_1, a_2, a_3, \cdots, a_{n-1}$ in order and since each column contains a_0 exactly one time, we easily see that each of the n columns $0, 1, 2, 3, \cdots, n-1$ will contain each of the lower case $a_0, a_1, \cdots, a_{n-1}$ exactly one time.

We now assign $A_0, A_1, A_2, \cdots, A_{n-1}$ to the n squares of the first column (which we call column 0) by the following rule.

Of course, A_0 has already been assigned to square $(0,0)$.

For each $1 \leq k \leq n-1$, we assign to the square $(k,0)$ the upper case letter $A_{(kt)\text{mod } n}$. As always, $(k,t) \bmod n$ is the remainder when kt is divided by n. Since (t,n) are relatively prime and$1 \leq k \leq n-1$, we see that $(kt) \bmod n \in \{1,2,\cdots,n-1\}$.

We now shorten this notation and simply say that to each square $(k,0), 1 \leq k \leq n-1$, we assign A_{kt}. Since (t,n) are relatively prime, we see that $\{kt : k \in \{1,2,3,\cdots,n-1\}\} = \{1,2,3,\cdots,n-1\}$, From this and from the fact that A_0 has been assigned to square $(0,0)$ we see that the n squares of the first column (which we are calling column 0), will contain each of the letters $A_{0,}A_{1,}A_2, \cdots, A_{n-1}$ exactly one time. Of course, in the top row (which we are calling row 0) we have in order the letters $A_{0,}A_{1,}A_2, \cdots, A_{n-1}$. For each row k, where $k \in \{1,2,3,\cdots,n-1\}$ we start with the letter A_{kt} in the first square $(k,0)$ and we write the letters $A_{0,}A_{1,}A_2, \cdots, A_{n-1}$ in this order. For example, suppose $n = 7, k = 4$ and suppose $A_{kt} = A_3$ has been assigned to the first square $(4,0)$ in row 4. We now assign $A_3, A_4, A_5, A_6, A_{0,}A_{1,}A_2$ in this order to the $n = 7$ squares of row $k = 4$. That is, A_3 is assigned to square $(4,0)$, A_4 is assigned to square $(4,1)$, A_5 is assigned to square $(4,2)$, etc.

Since the first column (which we are calling column 0) contains all of the letters $A_{0,}A_{1,}A_2$, $\cdots, A_{n-1}$ and since the letters $A_{0,}A_{1,}A_2, \cdots, A_{n-1}$ remain in this order in each row, it is easy to see that each row will contain each of the letters $A_{0,}A_{1,}A_2, \cdots, A_{n-1}$ exactly one time and each column will contain each of the letters $A_{0,}A_{1,}A_2, \cdots, A_{n-1}$ exactly one time.

Of course, in the first row (which we call row 0), the pair (a_0, A_0) appears in the first square $(0,0)$. Let us now consider row k where $1 \leq k \leq n-1$.

Now A_{kt} appears in the first square $(k,0)$ of row k.

Also, a_0 appears in square (k, kl) of row k. Since $A_{0,}A_{1,}A_2, \cdots, A_{n-1}$ remain in order in each row $1 \leq k \leq n-1$ and since A_{kt} appears in square $(k,0)$ we see that $A_{kt+kl} = A_{k(t+e)}$ appears in square (k, kl), where $k(t+e)$ is computed by mod n arithmetic. We know by hypothesis that $(t+e, n)$ are relatively prime. Therefore, $k(t+e) \in \{1,2,\cdots,n-1\}$. Therefore, in square (k, kl) of row $k, 1 \leq k \leq n-1$, we have the ordered pair $\left(a_0, A_{k(t+e)}\right)$ where $A_{k(t+e)} \in \{A_{1,}A_2, \cdots, A_{n-1}\}$.

Now $\{k(t+l) : k \in \{1,2,\cdots,n-1\}\} = \{1,2,3,\cdots,n-1\}$ since $(n, t+e)$ are relatively prime.

Also, (a_0, A_0) appears in square $(0,0)$. From this, we see that all of the ordered pairs $(a_0, A_0), (a_0, A_1), (a_0, A_2), (a_0, A_3), \cdots, (a_0, A_{n-1})$ are represented exactly one time on the $n \times n$ matrix that we are dealing with.

Since $a_0, a_1, a_2, \cdots, a_{n-1}$ in each row remain in this order and since $A_{0,}A_{1,}A_2, \cdots, A_{n-1}$ in each row remain in this order and since all of the ordered pairs $(a_0, A_0), (a_0, A_1), (a_0, A_2)$ $(a_0, A_3), \cdots, (a_0, A_{n-1})$ are represented one time each on the $n \times n$ matrix, we see that all of the ordered pairs $(r,s) \in \{a_0, a_1, a_2, \cdots, a_{n-1}\} \times \{A_{0,}A_{1,}A_2, \cdots, A_{n-1}\}$ are represented exactly one time each on the $n \times n$ matrix.

That is, for each $r \in \{a_0, a_1, a_2, \cdots, a_{n-1}\}$ and for each $s \in \{A_{0,}A_{1,}A_2, \cdots, A_{n-1}\}$ the ordered pair (r,s) is represented exactly one time on the $n \times n$ matrix. Remember now that $a_0, a_1, a_2, \cdots, a_{n-1}$ and $A_{0,}A_{1,}A_2, \cdots, A_{n-1}$ are permutations of $1,2,3,\cdots,n$.

For each ordered pair $(r,s) \in \{a_0, a_1, a_2, \cdots, a_{n-1}\} \times \{A_{0,}A_{1,}A_2, \cdots, A_{n-1}\}$, let us assign the number $(r,s)^{\#} = r + n(s-1)$ where r, s are the numerical values that have been assigned to the letters r, s.

Now $1 \le r \le (r,s)^{\#} = r + n(s-1) \le n + n(n-1) = n^2$. That is, $1 \le (r,s)^{\#} \le n^2$.

We now show that $(r,s)^{\#} = (\bar{r},\bar{s})^{\#}$ if and only if $r = \bar{r}$ and $s = \bar{s}$.

First, suppose $s \ne \bar{s}$ and by symmetry suppose $s < \bar{s}$. We show that $(r,s)^{\#} < (\bar{r},\bar{s})^{\#}$.

Now $(r,s)^{\#} = r + n(s-1)$ and $(\bar{r},\bar{s})^{\#} = \bar{r} + n(\bar{s}-1)$, we now show that $r + n(s-1) < \bar{r} + n(\bar{s}-1)$. This is equivalent to $r - \bar{r} < n(\bar{s} - s)$.

Since $r, \bar{r}, s, \bar{s} \in \{1, 2, 3, \cdots, n\}$ and $s < \bar{s}$, we see that $r - \bar{r} < n \le n(\bar{s} - s)$.

Therefore, $r - \bar{r} < n(\bar{s} - s)$.

If $s = \bar{s}$ then obviously, $(r,s)^{\#} = (\bar{r},\bar{s})^{\#}$ if and only if $r = \bar{r}$.

Therefore, $(r,s)^{\#} = (\bar{r},\bar{s})^{\#}$ if and only if $r = \bar{r}$ and $s = \bar{s}$.

From this we see that $\left\{(r,s)^{\#} : r, s \in \{1, 2, 3, \cdots, n\}\right\} = \{1, 2, 3, 4, \cdots, n^2\}$.

Let us now represent the entries of the $n \times n$ matrix that we are dealing with by (r_{ij}, s_{ij}) where r_{ij}, s_{ij} are the numerical values that are assigned to the ordered pair of square (i, j) where $i, j \in \{0, 1, 2, \cdots, n-1\}$. Thus, suppose (a_5, A_2) is the ordered pair assigned to the square $(2, 3)$ and suppose $a_5 = 7, A_2 = 4$. Then $(r_{23}, s_{23}) = (7, 4)$.

We also define the $n \times n$ matrix $(r_{ij}, s_{ij})^{\#}$, $i, j \in \{0, 1, 2, \cdots, n-1\}$, and show that this matrix is always a semi-magic square and is sometimes a magic square and is sometimes an extra magic square.

We now show that for each $t \in \{0, 1, 2, \cdots, n-1\}$

$(*) \sum_{i=0}^{n-1} (r_{ti}, s_{ti})^{\#} = \frac{n}{2}(n^2+1)$ and $(**) \sum_{i=0}^{n-1} (r_{it}, s_{it})^{\#} = \frac{n}{2}(n^2+1)$.

$(*), (**)$ mean that the sums of the n numbers in row t and column t respectively equal $\frac{n}{2}(n^2+1)$.

Now, $\sum_{i=0}^{n-1} (r_{ti}, s_{ti})^{\#} = \sum_{i=0}^{n-1} (r_{ti} + n(s_{ti} - 1)) = \sum_{i=0}^{n-1} r_{ti} + n \cdot \sum_{i=0}^{n-1} (s_{ti} - 1) = (***)$.

Now $\{r_{t0}, r_{ti}, r_{t2}, \cdots, r_{t,n-1}\} = \{1, 2, 3, \cdots, n\}$ since each row t contains all of the letters $a_0, a_1, a_2, \cdots, a_{n-1}$ and $a_0, a_1, a_2, \cdots, a_{n-1}$ is a permutation of $1, 2, 3, \cdots, n$.

Also, $\{s_{t0}, s_{t1}, s_{t2}, \cdots, s_{tn-1},\} = \{1, 2, 3, \cdots, n\}$ since each row t contain all of the letters $A_{0,}A_{1,}A_2, \cdots, A_{n-1}$ and $A_{0,}A_{1,}A_2, \cdots, A_{n-1}$ is a permutation of $1, 2, 3, \cdots, n$.

Thus,

$$\begin{aligned}
(***) &= (1 + 2 + \cdots + n) + n(0 + 1 + 2 + \cdots + (n-1)) \\
&= \frac{n}{2}(n+1) + n\left(\frac{n-1}{2}\right)(n) \\
&= \frac{n}{2}\left[n + 1 + n^2 - n\right] \\
&= \frac{n}{2}\left(n^2 + 1\right).
\end{aligned}$$

Likewise $(**)$ is true for the same reasons. Thus, from $(*), (**)$ we see that the $n \times n$ matrix become a semi-magic square when $(r_{ij}, s_{ij})^{\#}$ are assigned to squares (i, j). Instead of using the code $(r,s)^{\#} = r + n(s-1)$ we can also substitute the code $(r,s)^{\#} = s + n(r-1)$. This new code will always give a semi-magic $n \times n$ square when n is odd.

5 Using the Algorithm to Create Magic Squares

In the algorithm of Section 4 let $t = 1, e = 1$. Of course, (t, n) and (e, n) are relatively prime. Also, $(t + e, n) = (2, n)$ are relatively prime since n is odd.

As always, (a_0, A_0) is assigned to square $(0, 0)$.

In each row $k, 1 \leq k \leq n - 1$, we assign a_0 to square $(k, kl) = (k, k)$. This means that a_0 is assigned to each of the diagonal squares $(0, 0), (1, 1), (2, 2), \cdots, (n - 1, n - 1)$. Also, for each $1 \leq k \leq n-1$, we assign to the square $(k, 0)$ the upper case letter $A_{kt} = A_k$. Thus, in the first column we are assigning A_0 to square $(0, 0)$ and we assign in order $A_1, A_2, A_3, \cdots, A_{n-1}$ to the squares $(1, 0), (2, 0), (3, 0), \cdots, (n - 1, 0)$.

Now, the second diagonal of the $n \times n$ matrix consists of the squares $(0, n - 1), (1, n - 2), (2, n - 3), \cdots, (n - 2, 1), (n - 1, 0)$. That is, the second diagonal consists of the squares $(k, n - k - 1), 0 \leq k \leq n - 1$. Since A_k is assigned to square $(k, 0), 0 \leq k \leq n - 1$, we see that $A_{k+(n-k-1)} = A_{n-1}$ is assigned to each of the squares of the second diagonal.

Thus, in summary, a_0 is assigned to each of the n squares of the first diagonal and A_{n-1} is assigned to each of the n squares of the second diagonal.

From Section 4, we know that each of the ordered pairs $(r, s) \in \{a_0, a_1, a_2, \cdots, a_{n-1}\} \times \{A_0, A_1, A_2, \cdots, A_{n-1}\}$ is represented exactly one time on the $n \times n$ matrix.

Therefore, we can easily see that the n squares of the first diagonal will contain each of the ordered pairs $(a_0, A_0), (a_0, A_1), (a_0, A_2), \cdots, (a_0, A_{n-1})$ exactly one time.

Also, we easily see that the n squares of the second diagonal will contain each of the ordered pairs $(a_0, A_{n-1}), (a_1, A_{n-1}), (a_2, A_{n-1}), \cdots, (a_{n-1}, A_{n-1})$ exactly one time.

We now let $a_0, a_1, a_2, \cdots, a_{n-1}$ be any permutation of $1, 2, 3, \cdots, n$ subject to the one condition $a_0 = \frac{n+1}{2}$.

We also let $A_0, A_1, A_2, \cdots, A_{n-1}$be any permutation of $1, 2, 3, \cdots, n$ subject to the one condition $A_{n-1} = \frac{n+1}{2}$.

Of course, the algorithm of Section 4 will always produce a $n \times n$ semi-magic square when n is odd.

We now show that the $n \times n$ semi-magic square that we have just defined is also a magic square where the sum of the n entries in each of the two main diagonals equals $\frac{n}{2}(n^2 + 1)$. Now the sum of the n entries in the first diagonal squares $(0, 0), (1, 1), (2, 2), \cdots, (n - 1, n - 1)$ equals

$$
\begin{aligned}
\sum_{i=0}^{n-1} (a_0, A_i)^{\#} &= \sum_{i=0}^{n-1} (a_0 + n(A_i - 1)) \\
&= \left(\sum_{i=0}^{n-1} a_0 \right) + n \cdot \sum_{i=0}^{n-1} (A_i - 1) \\
&= n \cdot a_0 + n(0 + 1 + 2 + \cdots + n - 1) \\
&= n\left(\frac{n+1}{2}\right) + \frac{n^2}{2}(n - 1) \\
&= \frac{n}{2}\left[n + 1 + n^2 - n\right] \\
&= \frac{n}{2}\left(n^2 + 1\right).
\end{aligned}
$$

Also, the sum of the entries in the second diagonal squares $(0,n-1), (1,n-2), (2,n-3), \cdots, (n-1,0)$ equals

$$\begin{aligned}
\sum_{i=0}^{n-1} (a_i, A_{n-1})^{\#} &= \sum_{i=0}^{n-1} (a_i + n(A_{n-1} - 1)) \\
&= \sum_{i=0}^{n-1} a_i + n \cdot \sum_{i=0}^{n-1} (A_{n-1} - 1) \\
&= (1 + 2 + \cdots + n) + n \cdot \sum_{i=0}^{n-1} \left(\frac{n+1}{2} - 1 \right) \\
&= \frac{n}{2}(n+1) + n \left(\frac{n-1}{2} \right)(n) \\
&= \frac{n}{2}\left[n + 1 + n^2 - n\right] \\
&= \frac{n}{2}\left(n^2 + 1\right).
\end{aligned}$$

Thus, we have a magic $n \times n$ square when n is odd.

6 An Observation

In the Section 4 algorithm we know that for $1 \le k \le n-1, a_0$ is assigned to square (k, kl) where $kl \in \{1, 2, 3, \cdots, n-1\}$ and kl is computed mod n.

Thus, for $1 \le k \le n-1$ we have $a_{-kl} = a_{(n-l)k}$ assigned to square $(k, 0)$ where $-kl, n-l$, and $(n-l)k$ are computed mod n. Letting $n - l = \bar{l}$ we see that for $1 \le k \le n-1$ we have $a_{\bar{l}k}$ and A_{tk} assigned to square $(k, 0)$ where $(t, n), (\bar{l}, n), (t - \bar{l}, n)$ are all relatively prime. As always (a_0, A_0) is assigned to square $(0, 0)$. Thus, we have an alternative way of defining the $n \times n$ semi-magic square when n is odd.

7 A Specific Example of the Algorithm

In the Section 4 algorithm, we now let $n = 5, l = 2, t = 2, l + t = 4$. This leads to the following 5×5 matrix.

(a_0, A_0)	(a_1, A_1)	(a_2, A_2)	(a_3, A_3)	(a_4, A_4)
(a_3, A_2)	(a_4, A_3)	(a_0, A_4)	(a_1, A_0)	(a_2, A_1)
(a_1, A_4)	(a_2, A_0)	(a_3, A_1)	(a_4, A_2)	(a_0, A_3)
(a_4, A_1)	(a_0, A_2)	(a_1, A_3)	(a_2, A_4)	(a_3, A_0)
(a_2, A_3)	(a_3, A_4)	(a_4, A_0)	(a_0, A_1)	(a_0, A_2)

Fig. 2. A 5×5 matrix.

We now let $(a_0, a_1, a_2, a_3, a_4) = (3, 5, 1, 2, 4)$ and $(A_0, A_1, A_2, A_3, A_4) = (2, 1, 5, 4, 3)$.

Using the code $(r, s)^{\#} = r + n(s-1) = r + 5(s-1)$ with the 5×5 matrix of Fig. 2 we have the 5×5 semi-magic square of Fig. 3. The sum of the 5 entries in each row and in such column will be $\frac{5}{2}(5^2 + 1) = 65$.

8	5	21	17	14
22	19	13	10	1
15	6	2	24	18
4	23	20	11	7
16	12	9	3	25

Fig. 3 A 5×5 Semi-Magic Square

It turns out that Fig. 3 is also a magic square and an extra magic square. We deal with extra magic squares in Sections 9-11.

8 A 5×5 Magic Square

We let the reader use Section 5 to create a 5×5 Magic Square by letting $t = e = 1, n = 5, a_0 = \frac{n+1}{2} = 3, A_4 = \frac{n+1}{2} = 3$.

The reader can try different combinations from $\{a_1, a_2, a_3, a_4\} = \{1, 2, 4, 5\}$ and $\{A_0, A_1, A_2, A_3\} = \{1, 2, 4, 5\}$. All combinations will lead to Magic 5×5 Square.

9 Generalizing the Algorithm of Section 4

We can generalize the algorithm of Section 4 by replacing the sequence $a_0, a_m, a_{2m}, \cdots, a_{(n-1)m}$ for $a_1, a_2, a_3, \cdots, a_{n-1}$ and replacing the sequence $A_0, A_{\bar{m}}, A_{2\overline{m}}, \cdots, A_{(n-1)\overline{m}}$ for $A_0, A_1, A_2, \cdots, A_{n-1}$ where $m, \overline{m} \in \{1, 2, 3, \cdots, n-1\}$ and $(m, n), (\overline{m}, n)$ are relatively prime. The main ideas of Section 4 remain unchanged in this generalization. For example, A_{kt} goes in each square $(k, 0), k \in \{0, 1, 2, \cdots, n-1\}$, and a_0 goes in each square $(k, kl), k \in \{0, 1, 2, \cdots, n-1\}$, where $(t, n), (e, n)$ are relatively prime. Instead of requiring that $(t + e, n)$ be relatively prime we require that $(t + \overline{m}e, n)$ be relatively prime.

10 Extra Magic Squares

As in Section 5, the first main diagonal consists of the squares $(k, k), k \in \{0, 1, 2, \cdots, n-1\}$, and the second main diagonal consists of the squares $(k, n-1-k), k \in \{0, 1, 2, \cdots, n-1\}$. For each fixed $a \in \{0, 1, 2, \cdots, n-1\}$ we define a generalized first type diagonal D_a as $D_a = \{(k, k+a) : k \in \{0, 1, 2, \cdots, n-1\}\}$. Also, for each fixed $a \in \{0, 1, 2, \cdots, n-1\}$ we define a generalized second type diagonal $\overline{D}_a$ as $\overline{D}_a = \{(k, n-1-k+a) : k \in \{0, 1, 2, \cdots, n-1\}\}$. All operations use mod n arithmetic. A semi-magic $n \times n$ square is called a strong magic $n \times n$ square if the sum of the n entries on each generalized first type diagonal equals $\frac{n}{2}(n^2+1)$ and the sum of the n entries on each generalized second type diagonal equals $\frac{n}{2}(n^2+1)$.

11 Creating Extra Magic $n \times n$ Squares when n is Odd

If the average value of the a_i's on each generalized diagonal equals $\frac{n+1}{2}$ and the average value of the A_i's on each generalized diagonal equals $\frac{n+1}{2}$ and the sum of the n entries on each generalized diagonal will equal

$$\begin{aligned}
\sum_{(i,j)\in \text{diagonal}} (r_{ij}, s_{ij})^{\#} &= \sum_{(i,j)\in \text{ diagonal}} (r_{ij} + n(s_{ij}-1)) \\
&= \left(\sum r_{ij}\right) + n \cdot \left(\sum s_{ij}\right) - n^2 \\
&= n\left(\frac{n+1}{2}\right) + n^2\left(\frac{n+1}{2}\right) - n^2 \\
&= \frac{n}{2}\left[n+1+n^2+n-2n\right] \\
&= \frac{n}{2}\left(n^2+1\right).
\end{aligned}$$

One way to have the average value of the a_i's and the average value of the A_i's on each generalized diagonal equal to $\frac{n+1}{2}$ is to have each of $a_0, a_1, a_2, \cdots, a_{n-1}$ and to have each of $A_0, A_1, A_2, \cdots, A_{n-1}$ appear on each generalized diagonal. It is often possible to do this and we consider this first. As always, all arithmetic is mod n.

First, we observe that if each of the two main diagonals contain all of the letters $a_0, a_1, a_2, \cdots, a_{n-1}$ and contain all of the letters $A_0, A_1, A_2, \cdots, A_{n-1}$ then all of the generalized diagonals will also contain all of the letters a_0, a_1, a_{n-1} and contain all of the letters A_0, A_1, A_{n-1}.

For each row $k \in [0, 1, 2, \cdots, n-1]$ square $(k, 0)$ will contain A_{kt} and square (k, ek) will contain a_0. Therefore, the first main diagonal square (k, k) will contain $A_{kt+k} = A_{k(t+1)}$ and will contain $a_{0+k-kl} = a_{k(1-e)} = a_{k(n+1-e)}$. Therefore, the first main diagonal will contain all of $a_0, a_1, a_2, \cdots, a_{n-1}$ if $(e-1, n)$ are relatively prime and will contain all of $A_0, A_1, A_2, \cdots, A_{n-1}$ if $(t+1, n)$ are relatively prime.

Also, the second main diagonal square $(k, n-1-k)$ will contain $A_{kt+n-1-k} = A_{n-1+k(t-1)}$ and will contain $a_{0+n-1-k-kl} = a_{n-1-k(e+1)}$.

Therefore, the second main diagonal will contain all of $a_0, a_1, a_2, \cdots, a_{n-1}$ if $(e+1, n)$ are relatively prime and contain all of $A_0, A_1, A_2, \cdots, A_{n-1}$ if $(t-1, n)$ are relatively prime.

Recall that $(t, n), (l, n), (t+l, n)$ must be relatively prime in order to have a semi-magic square. If we also have $(t+1, n), (t-1, n), (l+1, n), (l-1, n)$ are relatively prime, then we have an extra magic square. This is easy to do when $n \geq 5$ and n is prime.

Next, suppose (a or b) is true and (c or d) is true in addition to $(t, n), (l, n), (t+l, n)$ are relatively prime.

a. $(t+1, n)$ are relatively prime and $n \nmid t-1$.

b. $(t-1, n)$ are relatively prime and $n \nmid t+1$.

c. $(l+1, n)$ are relatively prime and $n \nmid l-1$.

d. $(l-1, n)$ are relatively prime and $n \nmid l+1$.

It is now easy to use the idea stated at the beginning of this section to create extra magic $n \times n$ squares. We let the reader do this for himself when $n = 9, l = 2, t = 5, l - t = 7$. When the reader fills in the (a_{ij}, A_{ij})'s of the 9×9 matrix, he will observe the following.

Each of the generalized 1st diagonals will contain each of the letters $a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$.

Each of the generalized 1st diagonals will contain $A_0, A_3, A_6, A_0, A_3, A_6, A_0, A_3, A_6$ or $A_0, A_4, A_7, A_1, A_4, A_7, A_1, A_4, A_7$ or $A_2, A_5, A_8, A_2, A_5, A_8, A_2, A_5, A_8$.

Also, each of the generalized 2nd diagonals will contain each of the letters $A_0, A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8$. Also, each of the generalized 2nd diagonals will contain $a_0, a_3, a_6, a_0, a_3, a_6, a_0, a_3, a_6$ or $a_1, a_4, a_7, a_1, a_4, a_7, a_1, a_4, a_7$ or $a_2, a_5, a_8, a_2, a_5, a_8, a_2, a_5, a_8$. We now choose $a_0 = 1, a_3 = 5, a_6 = 9$ and $a_1 = 2, a_4 = 6, a_7 = 7$ and $a_2 = 3, a_5 = 4, a_8 = 8$.

Also, we choose $A_0 = 1, A_3 = 5, A_6 = 9$ and $A_1 = 2, A_4 = 6, A_7 = 7$ and $A_2 = 3, A_5 = 4, A_8 = 8$.

We now observe that

$$\begin{aligned}
\frac{a_0 + a_3 + a_6}{3} &= \frac{a_1 + a_4 + a_7}{3} \\
&= \frac{a_2 + a_5 + a_8}{3} \\
&= \frac{A_0 + A_3 + A_6}{3} \\
&= \frac{A_1 + A_4 + A_7}{3} \\
&= \frac{A_2 + A_5 + A_8}{3} \\
&= \frac{n+1}{2} \\
&= 5.
\end{aligned}$$

Of course, $\sum\limits_{i=0}^{8} \frac{a_i}{9} = \sum\limits_{i=0}^{8} \frac{A_i}{9} = \frac{n+1}{2} = 5$.

Therefore, when we use the code $(r,s)^{\#} = r + n\,(s-1) = r + 9\,(s-1)$ we will have an extra magic 9×9 square. This example illustrates the general pattern when (a or b) and (c or d) are true. As a project the reader can also consider $n = 15, l = t = 2$.

12 An Extra Magic 5×5 Square

We let $n = 5, l = t = 2, e + t = 4$. We see that $(l,n), (t,n), (l+t,n), (t-1,n), (t+1,n)$, $(l-1,n), (l+1,n)$ are all relatively prime.

This gives the extra magic 5×5 square of Figs. 2, 3 when we choose $(a_0, a_1, a_2, a_3, a_4) = (3,5,1,2,4)$ and $(A_0, A_1, A_2, A_3, A_4) = (2,1,5,4,3)$. Of course, any permutations a_0, a_1, a_2, a_3, a_4 and A_0, A_1, A_2, A_3, A_4 of $1,2,3,4,5$ will give an extra magic 5×5 square and the reader might like to try a few.

The generalized algorithm of Section 8 can be used to create far more extra magic 5×5 square.

In a subsequent paper, we show that the magic 5×5 square of Fig. 2 is also extremely magic. This means that it has 120 standard five element magic subsets whose sum is 65. These include the five rows, five columns, and the ten generalized diagonals.

13 Some Concluding Remarks

Suppose we have a $n \times n$ semi-magic square.

First, suppose $a_{ki_1}, a_{ki_2}, \cdots, a_{ki_m}$ and $a_{li_1}, a_{li_2}, \cdots, a_{li_m}$ are m entries in rows k, l of this $n \times n$ semi-magic square where $1 \leq k < l \leq n$ and $1 < i_1 < i_2 < \cdots < i_m \leq n$.

If $\sum_{j=1}^{m} a_{kij} = \sum_{j=1}^{m} a_{lij}$ then we can interchange $(a_{ki_1}, a_{k‘i_2}, \cdots, a_{ki_m})$ and $(a_{li_1}, a_{li_2}, a_{li_m})$ and still have a $n \times n$ semi-magic square.

Likewise, we can do the same thing for m entries, in columns $k < l$ of a $n \times n$ semi-magic square.

Thus, in particular we can interchange any two rows and interchange any two columns of a $n \times n$ semi-magic square, and still have a semi-magic square.

Finally, we mention that we must add a new technique in order to define a $n \times n$ semi-magic square or a $n \times n$ magic square when n is an even positive integer. We also note that there does not exist a semi-magic 2×2 square and this is one difficulty in defining semi-magic and magic $n \times n$ square when n is even.