Magic Hexagon Puzzles

Arthur Holshouser
3600 Bullard St.
Charlotte, NC,
USA

Harold Reiter
Department of Mathematics,
University of North Carolina Charlotte,
Charlotte, NC 28223, USA
hbreiter@uncc.edu

1 Abstract

We first state a magic hexagon puzzle, [1] and we later state all of its solutions in a lexicographical order. We also state a very difficult collection of puzzles that requires any solution i to be transformed into any other solution j by using at most six of the non-trivial transformations that we define. This very difficult puzzle has a reasonably easy solution when solution i is first transformed in at most 3 steps into the main (or central) solution and then the main (central) solution is transformed into solution j in at most 3 steps. The centralized nature of these puzzles is very different from what one usually sees in a puzzle.

2 Introduction

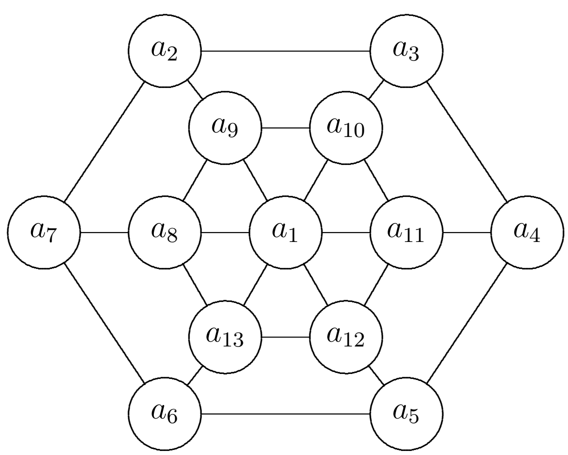

Fig. 1 A Magic Hexagon Puzzle, [1]

In this Magic Hexagon Puzzle, [1], we are required to distribute the numbers 1 through 13 in the 13 circles so that the sum of the five entries in each of the three lines through the center and the sum of the six entries in each of the two hexagons are identical.

Suppose k is the constant sum and suppose we have a solution with a_1 in the center, a_2

through a_7 in the outside hexagon and a_8 through a_{13} in the inner hexagon. Now $1+2+3+\cdots+13=91$. Therefore, $2k+a_1=91$. Also, $3a_1+(a_2+a_3+\cdots+a_{13})=3k$ since there are three lines through a_1. Therefore, $2a_1+(a_1+a_2+\cdots+a_{13})=2a_1+91=3k$. From $2k+a_1=91, 3k-2a_1=91$ we compute $k=39, a_1=13$.

In Fig. 1, we note that each pair $(a_2,a_5), (a_3,a_6), (a_4,a_7), (a_8,a_{11}), (a_9,a_{12})$ and (a_{10},a_{13}) appears in exactly one line and in exactly one hexagon.

Therefore, we can lump the numbers in each of these pairs together and consider the following magic 2×3 checkerboard.

a_2, a_5	a_3, a_6	a_4, a_7
a_9, a_{12}	a_{10}, a_{13}	a_8, a_{11}

Fig. 2. A Magic 2×3 checkerboard.

In the 2×3 Magic Checkerboard form of the Magic Hexagon Puzzle, we must do the following. (1) We distribute the numbers from the set $\{1,2,\cdots,12\}$ so that each square has two numbers. (2) We require the six numbers in each row to add up to 39 and (3) we require the four number in each column to add up to 26.

As in any magic checkerboard, a trivial magic preserving transformation is to interchange the two rows or to interchange any two columns. In this magic 2×3 checkerboard, a second trivial magic preserving transformation is the following. Suppose the sum of the two numbers a_i+a_j in one square equals the sum of the two numbers a_k+a_l in a second square. We can then interchange the two numbers $\{a_i,a_j\}$ in one square with the two numbers $\{a_k,a_l\}$ in the second square. The transitive closure (i.e., the connected components) of these two combined trivial transformations will partition the set of all of the 2×3 magic checkerboards into equivalency classes. In this note, the following notation is equivalent to the notation of Fig. 2:

$$\begin{bmatrix} (a_2,a_5) & (a_3,a_6) & (a_4,a_7) \\ (a_9,a_{12}) & (a_{10},a_{13}) & (a_8,a_{11}) \end{bmatrix}.$$

As always in matrix notation, the six squares of Fig. 2 or the six parenthesis in the equivalent matrix notation are denoted by $S_{11}, S_{12}, S_{13}, S_{21}, S_{22}, S_{23}$ where S_{ij} is the square in the i^{th} row and j^{th} column.

3 The Non-Trivial Transformations

In this note, we use three types of non-trivial magic preserving transformations.

1. Suppose $a_i\in S_{11}, a_j\in S_{12}, a_k\in S_{13}, a_l\in S_{21}, a_m\in S_{22}, a_n\in S_{23}$ and $a_i+a_j+a_k=a_l+a_m+a_n$. Then we can interchange the two ordered triples (a_i,a_j,a_k) and (a_l,a_m,a_n).

2. Suppose $a_i\in S_{1x}, a_j\in S_{1y}, x\neq y, a_k\in S_{2x}, a_l\in S_{2y}$ and $a_i+a_j=a_k+a_l$. Then we can interchange (a_i,a_j) and (a_k,a_l).

3. Suppose $a_i\in S_{1x}, a_j\in S_{2x}, a_k\in S_{1y}, a_l\in S_{2y}$, where $x\neq y$, and $a_i+a_j=a_k+a_l$. Then we can interchange $\binom{a_i}{a_j}$ and $\binom{a_k}{a_l}$.

For completeness, we list a 4th non-trivial transformation. This 4th non-trivial transformation is not used in this note because it is not needed. Only transformations 1, 2, 3 are used in this note.

4. Suppose $a_i, a_j \in S_{1x}, a_k \in S_{1y}, x \neq y, a_l, a_m \in S_{2x}, a_n \in S_{2y}$ where $a_i \neq a_j, a_l \neq a_m$ and $a_i + a_j + a_k = a_l + a_m + a_n$. Then we can interchange the two ordered triples (a_i, a_j, a_k) and (a_l, a_m, a_n).

Examples of non-trivial transformations 1, 2, 3 are given many times in Section 7. Of course, each trivial and nontrivial transformation gives a new magic 2×3 checkerboard.

In this note, the trivial transformations are denoted by $[\ ] \overset{\circ}{\to} [\ ]$ and the non-trivial transformations are denoted by $[\ ] \to [\ ]$.

4 The Main (Central) Solution to the Puzzle

Fig. 3 shows the main (or central) solution to the Magic Checkerboard form of the puzzle.

1, 12	2, 11	3, 10
4, 9	5, 8	6, 7

Fig. 3. The Main Solution

We note that the sum of the two numbers in each square is 13. Since the sum of the two numbers in each square is 13, the second type of trivial transformation allows us to interchange the two numbers in any one of these squares with the two number in any other square. Thus, we have $6! = 720$ main solutions. Of course, interchanging rows and columns is redundant. These 720 main solutions give us enormous freedom when we later solve the Difficult Puzzles of Section 6.

5 Finding All the Solutions to the Puzzle

In Fig. 2, let us consider the sum (or total) matrix

$$\begin{bmatrix} a_2 + a_5, a_3 + a_6, a_4 + a_7 \\ a_9 + a_{12}, a_{10} + a_{13}, a_8 + a_{11} \end{bmatrix} = \begin{bmatrix} T_{11}, T_{12}, T_{13} \\ T_{21}, T_{22}, T_{23} \end{bmatrix}.$$

Of course, $T_{11} + T_{12} + T_{13} = T_{21} + T_{22} + T_{23} = 39, T_{11} + T_{21} = T_{12} + T_{22} = T_{13} + T_{23} = 26$. Therefore, $T_{13}, T_{21}, T_{22}, T_{23}$ can be computed from T_{11}, T_{12}.

In Section 7, we list all of the possible matrices $\begin{bmatrix} T_{11}, T_{12}, T_{13} \\ T_{21}, T_{22}, T_{23} \end{bmatrix}$ in a lexicographical order and we state all of the solutions to the 2×3 checkerboard puzzle for each matrix $\begin{bmatrix} T_{11}, T_{12}, T_{13} \\ T_{21}, T_{22}, T_{23} \end{bmatrix}$ up to the equivalency classes of the two trivial transformations defined in Section 2.

In this lexicographical order, we let $T_{11} \geq T_{12} \geq T_{13}, T_{21} \leq T_{22} \leq T_{23}$ where $T_{21} = 26 - T_{11}, T_{22} = 26 - T_{12}, T_{23} = 26 - T_{13}$. Since $\frac{T_{11}+T_{12}+T_{13}}{3} = \frac{T_{21}+T_{22}+T_{23}}{3} = 13$ we know that $T_{11} \geq 13$ and $T_{21} \leq 13$.

For each matrix $\begin{bmatrix} T_{11}, T_{12}, T_{13} \\ T_{21}, T_{22}, T_{23} \end{bmatrix}$ and for each solution $\begin{bmatrix} (a_2, a_5), (a_3, a_6), (a_4, a_7) \\ (a_9, a_{12}), (a_{10}, a_{13}), (a_8, a_{11}) \end{bmatrix}$, we can transform this solution using the two trivial transformations of Section 2 to create all of the solutions in this same equivalency class. So essentially what we will be doing in Section 7 is listing one representative from each equivalency class.

A few of the matrices $\begin{bmatrix} T_{11}, T_{12}, T_{13} \\ T_{21}, T_{22}, T_{23} \end{bmatrix}$ are as follows: $\begin{bmatrix} 13, & 13, & 13 \\ 13, & 13, & 13 \end{bmatrix}$, $\begin{bmatrix} 14, & 14, & 11 \\ 12, & 12, & 15 \end{bmatrix}$, $\begin{bmatrix} 14, & 13, & 12 \\ 12, & 13, & 14 \end{bmatrix}$, $\begin{bmatrix} 15, & 15, & 9 \\ 11, & 11, & 17 \end{bmatrix}$, $\begin{bmatrix} 15, & 14, & 10 \\ 11, & 12, & 16 \end{bmatrix}$. $\begin{bmatrix} 15, & 13, & 11 \\ 11, & 13, & 15 \end{bmatrix}, \ldots$.

Note that in many of these matrices we are taking care of two cases in one. For example, when we get to $\begin{bmatrix} 15, & 12, & 12 \\ 11, & 14, & 14 \end{bmatrix}$ or $\begin{bmatrix} 17, & 11, & 11 \\ 9, & 15, & 15 \end{bmatrix}$ in the list we do not have to list these since they are the same as $\begin{bmatrix} 14, & 14, & 11 \\ 12, & 12, & 15 \end{bmatrix}$ and $\begin{bmatrix} 15, & 15, & 9 \\ 11, & 11, & 17 \end{bmatrix}$.

In Section 8, we mention another way of systematically computing all of the solutions to the 2×3 Magic Checkerboard Puzzle.

6 A Collection of Very Difficult Puzzles

Suppose $\begin{bmatrix} (a_2, a_5), (a_3, a_6), (a_4, a_7) \\ (a_9, a_{12}), (a_{10}, a_{13}), (a_8, a_{11}) \end{bmatrix}$ and $\begin{bmatrix} (\bar{a}_2, \bar{a}_5), (\bar{a}_3, \bar{a}_6), (\bar{a}_4, \bar{a}_7) \\ (\bar{a}_9, \bar{a}_{12}), (\bar{a}_{10}, \bar{a}_{13}), (\bar{a}_8, \bar{a}_{11}) \end{bmatrix}$ are any two solutions to the 2×3 Magic Checkerboard Puzzle.

The Difficult Puzzle requires us to transform the first solution into the second solution by using any sequence of trivial and nontrivial transformations 1, 2, 3 subject to the following restrictions. (1) We are allowed to use any number of trivial transformation, but (2) we can use at most six nontrivial transformations. Thus, the number six has two magic meanings in this note.

Our strategy for solving this puzzle is to first transform the first solution into the main (or central) solution of Fig. 3 using at most 3 of the nontrivial transformations 1, 2, 3. Then we transform the main (or central) solution into the second solution using at most 3 of the nontrivial transformations 1, 2, 3.

7 The List of All Solutions to the Magic 2×3 Checkerboard Puzzle and the Three Step Transformations of Each Solution to the Main (Central) Solution.

In this section, we list in Lexicographical order a group of 46 $\begin{bmatrix} T_{11}, T_{12}, T_{13} \\ T_{21}, T_{22}, T_{23} \end{bmatrix}$ matrices which are all of the $[T_{ij}]$ matrices that apply to the 2×3 Magic Checkerboard Puzzle. For each matrix $\begin{bmatrix} T_{11}, T_{12}, T_{13} \\ T_{21}, T_{22}, T_{23} \end{bmatrix}$, we list one representative $\begin{bmatrix} (a_2, a_5), (a_3, a_6), (a_4, a_7) \\ (a_9, a_{12}), (a_{10}, a_{13}), (a_8, a_{11}) \end{bmatrix}$ from each of the trivial transformation equivalency solution classes of the 2×3 Magic Checkerboard Puzzle for that $\begin{bmatrix} T_{11}, T_{12}, T_{13} \\ T_{21}, T_{22}, T_{23} \end{bmatrix}$. This was explained in Section 2, 5. These representatives are called (a), (b), (c), (d), $\cdots$ and we know that no solution (i) can be transformed into another solution (j) by using a sequence of trivial transformations. As an example, consider the Group 2 $[T_{ij}]$ matrix $\begin{bmatrix} 14, & 14, & 11 \\ 12, & 12, & 15 \end{bmatrix}$. The trivial transformations allow us to interchange rows and interchange columns. We can also interchange the two numbers in the two 14 squares and interchange the two numbers in the two 12 squares.

The representatives (a), (b), (c), (d), $\cdots$ after each $[T_{ij}]$ matrix were computed by systematic calculations.

After each solution (a), (b), (c), (d),$\cdots$ we immediately transform that solution into the Main (or Central) Solution by using at most 3 of the nontrivial transformations 1, 2, 3 defined in Section 3. These 3 step transformations will give the complete solution to the collection of very difficult puzzle stated in Section 6.

Group 1 $\begin{bmatrix} 13, & 13, & 13 \\ 13, & 13, & 13 \end{bmatrix}$.

(a) $\begin{bmatrix} (12,1) & (11,2) & (10,3) \\ (9,4) & (8,5) & (7,6) \end{bmatrix}$.

Group 2 $\begin{bmatrix} 14, & 14, & 11 \\ 12, & 12, & 15 \end{bmatrix}$.

(a)

$$\begin{bmatrix} (10,4) & (8,6) & (9,2) \\ (7,5) & (11,1) & (12,3) \end{bmatrix} \overset{\circ}{\to} \begin{bmatrix} (8,6) & (10,4) & (9,2) \\ (7,5) & (11,1) & (12,3) \end{bmatrix} \to \begin{bmatrix} (8,6) & (10,2) & (9,4) \\ (7,5) & (11,3) & (12,1) \end{bmatrix} \to \begin{bmatrix} (8,5) & (10,3) & (9,4) \\ (7,6) & (11,2) & (12,1) \end{bmatrix}.$$

We note that the first transformation is the trivial interchange of $(10,4), (8,6)$ since $10+4 = 8+6$. The second transformation is the nontrivial interchange $\binom{4}{1}, \binom{2}{3}$since $4+1 = 2+3$. The last transformation is the nontrivial interchange $(6,2), (5,3)$ since $6+2 = 5+3$. Recall that $[\ ] \overset{\circ}{\to} [\ ]$ denotes a trivial transformation.

(b)

$$\begin{bmatrix} (8,6) & (9,5) & (7,4) \\ (10,2) & (11,1) & (12,3) \end{bmatrix} \overset{\circ}{\to} \begin{bmatrix} (9,5) & (8,6) & (7,4) \\ (10,2) & (11,1) & (12,3) \end{bmatrix} \to \begin{bmatrix} (9,5) & (8,4) & (7,6) \\ (10,2) & (11,3) & (12,1) \end{bmatrix} \to \begin{bmatrix} (9,4) & (8,5) & (7,6) \\ (10,3) & (11,2) & (12,1) \end{bmatrix}.$$

(c) $\begin{bmatrix} (12,2) & (8,6) & (10,1) \\ (7,5) & (9,3) & (11,4) \end{bmatrix} \overset{\circ}{\to} \begin{bmatrix} (8,6) & (12,2) & (10,1) \\ (7,5) & (9,3) & (11,4) \end{bmatrix} \to \begin{bmatrix} (8,6) & (12,1) & (10,2) \\ (7,5) & (9,4) & (11,3) \end{bmatrix} \to$

$\begin{bmatrix} (8,5) & (12,1) & (10,3) \\ (7,6) & (9,4) & (11,2) \end{bmatrix}$. (d) $\begin{bmatrix} (12,2) & (8,6) & (7,4) \\ (11,1) & (9,3) & (10,5) \end{bmatrix} \to \begin{bmatrix} (12,2) & (8,4) & (7,6) \\ (11,1) & (9,5) & (10,3) \end{bmatrix} \to$

$\begin{bmatrix} (12,1) & (8,5) & (7,6) \\ (11,2) & (9,4) & (10,3) \end{bmatrix}$. (e) $\begin{bmatrix} (10,4) & (12,2) & (8,3) \\ (11,1) & (7,5) & (9,6) \end{bmatrix} \overset{\circ}{\to} \begin{bmatrix} (12,2) & (10,4) & (8,3) \\ (11,1) & (7,5) & (9,6) \end{bmatrix} \to$

$\begin{bmatrix} (12,1) & (10,4) & (9,3) \\ (11,2) & (7,5) & (8,6) \end{bmatrix} \to \begin{bmatrix} (12,1) & (10,3) & (9,4) \\ (11,2) & (7,6) & (8,5) \end{bmatrix}$.

(f) $\begin{bmatrix} (10,4) & (12,2) & (6,5) \\ (11,1) & (9,3) & (8,7) \end{bmatrix} \overset{\circ}{\to} \begin{bmatrix} (12,2) & (10,4) & (6,5) \\ (11,1) & (9,3) & (8,7) \end{bmatrix} \to \begin{bmatrix} (12,1) & (10,3) & (6,7) \\ (11,2) & (9,4) & (8,5) \end{bmatrix}$.

Note in the last transformation that we are interchanging $(2,4,5), (1,3,7)$ since $2+4+5 = 1+3+7$.

(g) $\begin{bmatrix} (12,2) & (11,3) & (10,1) \\ (7,5) & (8,4) & (9,6) \end{bmatrix} \to \begin{bmatrix} (12,1) & (11,3) & (10,2) \\ (7,6) & (8,4) & (9,5) \end{bmatrix} \to \begin{bmatrix} (12,1) & (11,2) & (10,3) \\ (7,6) & (8,5) & (9,4) \end{bmatrix}$.

Group 3 $\begin{bmatrix} 14, & 13, & 12 \\ 12, & 13, & 14 \end{bmatrix}$. (a) $\begin{bmatrix} (12,2) & (7,6) & (9,3) \\ (11,1) & (8,5) & (10,4) \end{bmatrix} \to \begin{bmatrix} (12,1) & (7,6) & (9,4) \\ (11,2) & (8,5) & (10,3) \end{bmatrix}$.

(b) $\begin{bmatrix} (12,2) & (10,3) & (11,1) \\ (8,4) & (7,6) & (9,5) \end{bmatrix} \to \begin{bmatrix} (12,1) & (10,3) & (11,2) \\ (8,5) & (7,6) & (9,4) \end{bmatrix}$. (c) $\begin{bmatrix} (12,2) & (10,3) & (7,5) \\ (11,1) & (9,4) & (8,6) \end{bmatrix} \to$

$\begin{bmatrix} (12,1) & (10,3) & (7,6) \\ (11,2) & (9,4) & (8,5) \end{bmatrix}$. (d) $\begin{bmatrix} (11,3) & (12,1) & (7,5) \\ (10,2) & (9,4) & (8,6) \end{bmatrix} \to \begin{bmatrix} (11,2) & (12,1) & (7,6) \\ (10,3) & (9,4) & (8,5) \end{bmatrix}$, (e)

$\begin{bmatrix} (9,5) & (12,1) & (8,4) \\ (10,2) & (7,6) & (11,3) \end{bmatrix} \to \begin{bmatrix} (9,4) & (12,1) & (8,5) \\ (10,3) & (7,6) & (11,2) \end{bmatrix}$. (f) $\begin{bmatrix} (10,4) & (12,1) & (7,5) \\ (9,3) & (11,2) & (8,6) \end{bmatrix} \to$

$\begin{bmatrix} (10,3) & (12,1) & (7,6) \\ (9,4) & (11,2) & (8,5) \end{bmatrix}$.

Group 4 $\begin{bmatrix} 15, & 15, & 9 \\ 11, & 11, & 17 \end{bmatrix}$. (a) $\begin{bmatrix} (11,4) & (8,7) & (6,3) \\ (10,1) & (9,2) & (12,5) \end{bmatrix} \overset{\circ}{\to} \begin{bmatrix} (8,7) & (11,4) & (6,3) \\ (10,1) & (9,2) & (12,5) \end{bmatrix} \to$

$$\begin{bmatrix} (8,3) & (11,4) & (6,7) \\ (10,5) & (9,2) & (12,1) \end{bmatrix} \to \begin{bmatrix} (8,5) & (11,2) & (6,7) \\ (10,3) & (9,4) & (12,1) \end{bmatrix} \text{(b)} \begin{bmatrix} (12,3) & (8,7) & (5,4) \\ (10,1) & (9,2) & (11,6) \end{bmatrix} \to$$

$$\begin{bmatrix} (12,1) & (8,7) & (5,6) \\ (10,3) & (9,2) & (11,4) \end{bmatrix} \to \begin{bmatrix} (12,1) & (8,5) & (7,6) \\ (10,3) & (9,4) & (11,2) \end{bmatrix}. \text{(c)} \begin{bmatrix} (12,3) & (10,5) & (8,1) \\ (9,2) & (7,4) & (11,6) \end{bmatrix} \overset{\circ}{\to}$$

$$\begin{bmatrix} (10,5) & (12,3) & (8,1) \\ (9,2) & (7,4) & (11,6) \end{bmatrix} \to \begin{bmatrix} (10,1) & (12,3) & (8,5) \\ (9,6) & (7,4) & (11,2) \end{bmatrix} \to \begin{bmatrix} (10,3) & (12,1) & (8,5) \\ (9,4) & (7,6) & (11,2) \end{bmatrix}.$$

$$\text{(d)} \begin{bmatrix} (12,3) & (11,4) & (7,2) \\ (10,1) & (6,5) & (9,8) \end{bmatrix} \to \begin{bmatrix} (12,1) & (11,6) & (7,2) \\ (10,3) & (4,5) & (9,8) \end{bmatrix} \to \begin{bmatrix} (12,1) & (11,2) & (7,6) \\ (10,3) & (8,5) & (9,4) \end{bmatrix}.$$

$$\text{(e)} \begin{bmatrix} (12,3) & (11,4) & (8,1) \\ (9,2) & (6,5) & (10,7) \end{bmatrix} \overset{\circ}{\to} \begin{bmatrix} (11,4) & (12,3) & (8,1) \\ (9,2) & (6,5) & (10,7) \end{bmatrix} \to \begin{bmatrix} (11,2) & (12,5) & (8,1) \\ (9,4) & (6,3) & (10,7) \end{bmatrix} \to$$

$$\begin{bmatrix} (11,2) & (12,1) & (8,5) \\ (9,4) & (6,7) & (10,3) \end{bmatrix}. \text{(f)} \begin{bmatrix} (11,4) & (9,6) & (7,2) \\ (10,1) & (8,3) & (12,5) \end{bmatrix} \overset{\circ}{\to} \begin{bmatrix} (9,6) & (11,4) & (7,2) \\ (10,1) & (8,3) & (12,5) \end{bmatrix} \to$$

$$\begin{bmatrix} (9,4) & (11,6) & (7,2) \\ (10,3) & (8,1) & (12,5) \end{bmatrix} \to \begin{bmatrix} (9,4) & (11,2) & (7,6) \\ (10,3) & (8,5) & (12,1) \end{bmatrix}.$$

Group 5 $\begin{bmatrix} 15, & 14, & 10 \\ 11, & 12, & 16 \end{bmatrix}$. (a) $\begin{bmatrix} (8,7) & (11,3) & (9,1) \\ (6,5) & (10,2) & (12,4) \end{bmatrix} \to \begin{bmatrix} (8,5) & (11,2) & (9,4) \\ (6,7) & (10,3) & (12,1) \end{bmatrix}$.

In (a) we note that $(7,3,1),(5,2,4)$ can be interchanged since $7+3+1=5+2+4$. (b)

$$\begin{bmatrix} (9,6) & (11,3) & (8,2) \\ (10,1) & (7,5) & (12,4) \end{bmatrix} \to \begin{bmatrix} (9,10) & (7,3) & (8,2) \\ (6,1) & (11,5) & (12,4) \end{bmatrix} \overset{\circ}{\to} \begin{bmatrix} (9,10) & (8,2) & (7,3) \\ (6,1) & (11,5) & (12,4) \end{bmatrix} \to$$

$$\begin{bmatrix} (6,10) & (8,5) & (7,3) \\ (9,1) & (11,2) & (12,4) \end{bmatrix} \to \begin{bmatrix} (6,7) & (8,5) & (10,3) \\ (9,4) & (11,2) & (12,1) \end{bmatrix}. \text{(c)} \begin{bmatrix} (12,3) & (8,6) & (9,1) \\ (7,4) & (10,2) & (11,5) \end{bmatrix} \to$$

$$\begin{bmatrix} (12,7) & (2,6) & (11,1) \\ (3,4) & (10,8) & (9,5) \end{bmatrix} \to \begin{bmatrix} (12,1) & (2,6) & (11,7) \\ (9,4) & (10,8) & (3,5) \end{bmatrix} \to \begin{bmatrix} (12,1) & (2,11) & (6,7) \\ (9,4) & (10,3) & (8,5) \end{bmatrix}.$$

$$\text{(d)} \begin{bmatrix} (9,6) & (12,2) & (7,3) \\ (10,1) & (8,4) & (11,5) \end{bmatrix} \to \begin{bmatrix} (9,10) & (4,2) & (11,3) \\ (6,1) & (8,12) & (7,5) \end{bmatrix} \to \begin{bmatrix} (9,11) & (4,2) & (10,3) \\ (5,1) & (8,12) & (7,6) \end{bmatrix} \to$$

$$\begin{bmatrix} (9,4) & (11,2) & (10,3) \\ (5,8) & (1,12) & (7,6) \end{bmatrix}. \text{(e)} \begin{bmatrix} (8,7) & (12,2) & (6,4) \\ (10,1) & (9,3) & (11,5) \end{bmatrix} \to \begin{bmatrix} (10,7) & (3,2) & (6,11) \\ (8,1) & (9,12) & (4,5) \end{bmatrix} \to$$

$$\begin{bmatrix} (10,11) & (3,2) & (6,7) \\ (4,1) & (9,12) & (8,5) \end{bmatrix} \to \begin{bmatrix} (10,3) & (11,2) & (6,7) \\ (12,1) & (9,4) & (8,5) \end{bmatrix}. \text{(f)} \begin{bmatrix} (11,4) & (12,2) & (9,1) \\ (8,3) & (7,5) & (10,6) \end{bmatrix} \to$$

$$\begin{bmatrix} (11,4) & (12,1) & (9,2) \\ (8,3) & (7,6) & (10,5) \end{bmatrix} \to \begin{bmatrix} (11,2) & (12,1) & (9,4) \\ (8,5) & (7,6) & (10,3) \end{bmatrix}. \text{ (g) } \begin{bmatrix} (10,5) & (8,6) & (7,3) \\ (9,2) & (11,1) & (12,4) \end{bmatrix} \to$$

$$\begin{bmatrix} (10,3) & (8,6) & (7,5) \\ (9,4) & (11,1) & (12,2) \end{bmatrix} \to \begin{bmatrix} (10,3) & (8,5) & (7,6) \\ (9,4) & (11,2) & (12,1) \end{bmatrix}. \text{ (h) } \begin{bmatrix} (12,3) & (10,4) & (8,2) \\ (6,5) & (11,1) & (9,7) \end{bmatrix} \to$$

$$\begin{bmatrix} (6,3) & (10,11) & (7,2) \\ (12,5) & (4,1) & (9,8) \end{bmatrix} \to \begin{bmatrix} (2,3) & (10,11) & (7,6) \\ (12,9) & (4,1) & (5,8) \end{bmatrix} \to \begin{bmatrix} (2,11) & (10,3) & (7,6) \\ (12,1) & (4,9) & (5,8) \end{bmatrix}. \text{ (i)}$$

$$\begin{bmatrix} (12,3) & (9,5) & (8,2) \\ (7,4) & (11,1) & (10,6) \end{bmatrix} \to \begin{bmatrix} (4,3) & (9,11) & (10,2) \\ (7,12) & (5,1) & (8,6) \end{bmatrix} \to \begin{bmatrix} (4,9) & (3,11) & (10,2) \\ (1,12) & (5,7) & (8,6) \end{bmatrix} \to$$

$$\begin{bmatrix} (4,9) & (3,10) & (11,2) \\ (1,12) & (6,7) & (8,5) \end{bmatrix}. \text{ (j) } \begin{bmatrix} (10,5) & (12,2) & (6,4) \\ (8,3) & (11,1) & (9,7) \end{bmatrix} \to \begin{bmatrix} (8,5) & (11,2) & (6,7) \\ (10,3) & (12,1) & (9,4) \end{bmatrix}.$$

Group 6 $\begin{bmatrix} 15 & 13 & 11 \\ 11 & 13 & 15 \end{bmatrix}$ (a) $\begin{bmatrix} (12,3) & (8,5) & (9,2) \\ (10,1) & (7,6) & (11,4) \end{bmatrix} \to \begin{bmatrix} (12,1) & (8,5) & (9,4) \\ (10,3) & (7,6) & (11,2) \end{bmatrix}.$

$$\text{(b) } \begin{bmatrix} (12,3) & (11,2) & (6,5) \\ (10,1) & (9,4) & (8,7) \end{bmatrix} \to \begin{bmatrix} (12,1) & (11,2) & (6,7) \\ (10,3) & (9,4) & (8,5) \end{bmatrix}. \text{ (c) } \begin{bmatrix} (12,3) & (11,2) & (7,4) \\ (10,1) & (8,5) & (9,6) \end{bmatrix} \to$$

$$\begin{bmatrix} (12,1) & (11,2) & (7,6) \\ (10,3) & (8,5) & (9,4) \end{bmatrix}. \text{ (d) } \begin{bmatrix} (11,4) & (12,1) & (6,5) \\ (9,2) & (10,3) & (8,7) \end{bmatrix} \to \begin{bmatrix} (11,2) & (12,1) & (6,7) \\ (9,4) & (10,3) & (8,5) \end{bmatrix}. \text{ (e)}$$

$$\begin{bmatrix} (11,4) & (12,1) & (8,3) \\ (9,2) & (7,6) & (10,5) \end{bmatrix} \to \begin{bmatrix} (11,2) & (12,1) & (8,5) \\ (9,4) & (7,6) & (10,3) \end{bmatrix}. \text{ (f) } \begin{bmatrix} (10,5) & (12,1) & (7,4) \\ (8,3) & (11,2) & (9,6) \end{bmatrix} \to$$

$$\begin{bmatrix} (10,3) & (12,1) & (7,6) \\ (8,5) & (11,2) & (9,4) \end{bmatrix}.$$

Group 7 $\begin{bmatrix} 16 & 16 & 7 \\ 10 & 10 & 19 \end{bmatrix}$ (a) $\begin{bmatrix} (11,5) & (10,6) & (4,3) \\ (9,1) & (8,2) & (12,7) \end{bmatrix} \overset{\circ}{\to} \begin{bmatrix} (10,6) & (11,5) & (4,3) \\ (9,1) & (8,2) & (12,7) \end{bmatrix} \to$

$$\begin{bmatrix} (10,6) & (11,2) & (7,3) \\ (9,1) & (8,5) & (12,4) \end{bmatrix} \to \begin{bmatrix} (10,3) & (11,2) & (7,6) \\ (9,4) & (8,5) & (12,1) \end{bmatrix}. \text{ (b)}$$

$$\begin{bmatrix} (12,4) & (10,6) & (5,2) \\ (9,1) & (7,3) & (11,8) \end{bmatrix} \to \begin{bmatrix} (12,1) & (10,3) & (5,8) \\ (9,4) & (7,6) & (11,2) \end{bmatrix}. \text{(c)} \begin{bmatrix} (12,4) & (11,5) & (6,1) \\ (8,2) & (7,3) & (10,9) \end{bmatrix} \overset{\circ}{\to}$$

$$\begin{bmatrix} (11,5) & (12,4) & (6,1) \\ (8,2) & (7,3) & (10,9) \end{bmatrix} \to \begin{bmatrix} (11,2) & (12,4) & (9,1) \\ (8,5) & (7,3) & (10,6) \end{bmatrix} \to \begin{bmatrix} (11,2) & (12,1) & (9,4) \\ (8,5) & (7,6) & (10,3) \end{bmatrix}.$$

Group 8 $\begin{bmatrix} 16 & 15 & 8 \\ 10 & 11 & 18 \end{bmatrix}$ (a) $\begin{bmatrix} (9,7) & (11,4) & (5,3) \\ (8,2) & (10,1) & (12,6) \end{bmatrix} \to \begin{bmatrix} (9,8) & (1,4) & (5,12) \\ (7,2) & (10,11) & (3,6) \end{bmatrix} \to$

$$\begin{bmatrix} (9,4) & (1,8) & (5,12) \\ (11,2) & (10,7) & (3,6) \end{bmatrix} \to \begin{bmatrix} (9,4) & (1,12) & (5,8) \\ (11,2) & (10,3) & (7,6) \end{bmatrix}. \text{(b)} \begin{bmatrix} (12,4) & (9,6) & (5,3) \\ (8,2) & (10,1) & (11,7) \end{bmatrix} \to$$

$$\begin{bmatrix} (2,4) & (9,10) & (11,3) \\ (8,12) & (6,1) & (5,7) \end{bmatrix} \to \begin{bmatrix} (3,4) & (9,10) & (11,2) \\ (7,12) & (6,1) & (5,8) \end{bmatrix} \to \begin{bmatrix} (3,10) & (9,4) & (11,2) \\ (1,12) & (6,7) & (5,8) \end{bmatrix}.$$

(c) $\begin{bmatrix} (12,4) & (10,5) & (6,2) \\ (9,1) & (8,3) & (11,7) \end{bmatrix} \to \begin{bmatrix} (9,4) & (8,5) & (11,2) \\ (12,1) & (10,3) & (6,7) \end{bmatrix}$. (d) $\begin{bmatrix} (11,5) & (12,3) & (6,2) \\ (9,1) & (7,4) & (10,8) \end{bmatrix} \to$

$$\begin{bmatrix} (11,9) & (4,3) & (10,2) \\ (5,1) & (7,12) & (6,8) \end{bmatrix} \to \begin{bmatrix} (11,9) & (10,3) & (4,2) \\ (5,1) & (7,6) & (12,8) \end{bmatrix} \to \begin{bmatrix} (11,2) & (10,3) & (4,9) \\ (5,8) & (7,6) & (12,1) \end{bmatrix}.$$

(e) $\begin{bmatrix} (11,5) & (12,3) & (7,1) \\ (6,4) & (9,2) & (10,8) \end{bmatrix} \to \begin{bmatrix} (4,5) & (12,9) & (8,1) \\ (6,11) & (3,2) & (10,7) \end{bmatrix} \to \begin{bmatrix} (4,5) & (12,1) & (8,9) \\ (6,11) & (3,10) & (2,7) \end{bmatrix} \to$

$$\begin{bmatrix} (4,9) & (12,1) & (8,5) \\ (2,11) & (3,10) & (6,7) \end{bmatrix}.$$

Group 9 $\begin{bmatrix} 16 & 14 & 9 \\ 10 & 12 & 17 \end{bmatrix}$ (a) $\begin{bmatrix} (9,7) & (11,3) & (8,1) \\ (6,4) & (10,2) & (12,5) \end{bmatrix} \to \begin{bmatrix} (6,7) & (10,3) & (8,5) \\ (9,4) & (11,2) & (12,1) \end{bmatrix}$.

(b) $\begin{bmatrix} (10,6) & (11,3) & (7,2) \\ (9,1) & (8,4) & (12,5) \end{bmatrix} \to \begin{bmatrix} (7,6) & (11,3) & (10,2) \\ (12,1) & (8,4) & (9,5) \end{bmatrix} \to \begin{bmatrix} (7,6) & (11,2) & (10,3) \\ (12,1) & (8,5) & (9,4) \end{bmatrix}$.

(c) $\begin{bmatrix} (9,7) & (10,4) & (6,3) \\ (8,2) & (11,1) & (12,5) \end{bmatrix} \to \begin{bmatrix} (6,7) & (10,4) & (9,3) \\ (8,5) & (11,1) & (12,2) \end{bmatrix} \to \begin{bmatrix} (6,7) & (10,3) & (9,4) \\ (8,5) & (11,2) & (12,1) \end{bmatrix}$.

(d) $\begin{bmatrix} (12,4) & (9,5) & (8,1) \\ (7,3) & (10,2) & (11,6) \end{bmatrix} \to \begin{bmatrix} (12,1) & (9,5) & (8,4) \\ (7,6) & (10,2) & (11,3) \end{bmatrix} \to \begin{bmatrix} (12,1) & (9,4) & (8,5) \\ (7,6) & (10,3) & (11,2) \end{bmatrix}$.

(e) $\begin{bmatrix} (12,4) & (9,5) & (6,3) \\ (8,2) & (11,1) & (10,7) \end{bmatrix} \to \begin{bmatrix} (12,4) & (9,1) & (6,7) \\ (8,2) & (11,5) & (10,3) \end{bmatrix} \to \begin{bmatrix} (12,1) & (9,4) & (6,7) \\ (8,5) & (11,2) & (10,3) \end{bmatrix}$.

(f) $\begin{bmatrix} (11,5) & (12,2) & (6,3) \\ (9,1) & (8,4) & (10,7) \end{bmatrix} \to \begin{bmatrix} (11,2) & (12,5) & (6,3) \\ (9,4) & (8,1) & (10,7) \end{bmatrix} \to \begin{bmatrix} (11,2) & (12,1) & (6,7) \\ (9,4) & (8,5) & (10,3) \end{bmatrix}$.

(g) $\begin{bmatrix} (11,5) & (12,2) & (8,1) \\ (6,4) & (9,3) & (10,7) \end{bmatrix} \to \begin{bmatrix} (11,2) & (12,5) & (8,1) \\ (9,4) & (6,3) & (10,7) \end{bmatrix} \to \begin{bmatrix} (11,2) & (12,1) & (8,5) \\ (9,4) & (6,7) & (10,3) \end{bmatrix}$.

(h) $\begin{bmatrix} (10,6) & (12,2) & (5,4) \\ (7,3) & (11,1) & (9,8) \end{bmatrix} \to \begin{bmatrix} (7,6) & (11,2) & (5,8) \\ (10,3) & (12,1) & (9,4) \end{bmatrix}$.

Group 10 $\begin{bmatrix} 16 & 13 & 10 \\ 10 & 13 & 16 \end{bmatrix}$ (a) $\begin{bmatrix} (11,5) & (10,3) & (8,2) \\ (9,1) & (7,6) & (12,4) \end{bmatrix} \to \begin{bmatrix} (11,2) & (10,3) & (8,5) \\ (9,4) & (7,6) & (12,1) \end{bmatrix}$.

(b) $\begin{bmatrix} (10,6) & (11,2) & (7,3) \\ (9,1) & (8,5) & (12,4) \end{bmatrix} \to \begin{bmatrix} (10,3) & (11,2) & (7,6) \\ (9,4) & (8,5) & (12,1) \end{bmatrix}$. (c) $\begin{bmatrix} (10,6) & (12,1) & (7,3) \\ (8,2) & (9,4) & (11,5) \end{bmatrix} \to$

$\begin{bmatrix} (10,3) & (12,1) & (7,6) \\ (8,5) & (9,4) & (11,2) \end{bmatrix}$. (d) $\begin{bmatrix} (11,5) & (12,1) & (6,4) \\ (8,2) & (10,3) & (9,7) \end{bmatrix} \to \begin{bmatrix} (11,2) & (12,1) & (6,7) \\ (8,5) & (10,3) & (9,4) \end{bmatrix}$.

Group 11 $\begin{bmatrix} 17 & 17 & 5 \\ 9 & 9 & 21 \end{bmatrix}$ (a) $\begin{bmatrix} (11,6) & (10,7) & (3,2) \\ (8,1) & (5,4) & (12,9) \end{bmatrix} \overset{\circ}{\to} \begin{bmatrix} (10,7) & (11,6) & (3,2) \\ (8,1) & (5,4) & (12,9) \end{bmatrix} \to$

$\begin{bmatrix} (6,7) & (11,10) & (3,2) \\ (8,5) & (1,4) & (12,9) \end{bmatrix} \to \begin{bmatrix} (6,7) & (11,2) & (3,10) \\ (8,5) & (9,4) & (12,1) \end{bmatrix}$. (b) $\begin{bmatrix} (12,5) & (9,8) & (4,1) \\ (7,2) & (6,3) & (11,10) \end{bmatrix} \overset{\circ}{\to}$

$\begin{bmatrix} (9,8) & (12,5) & (4,1) \\ (7,2) & (6,3) & (11,10) \end{bmatrix} \to \begin{bmatrix} (9,8) & (12,1) & (4,5) \\ (7,2) & (10,3) & (11,6) \end{bmatrix} \to \begin{bmatrix} (9,4) & (12,1) & (8,5) \\ (7,6) & (10,3) & (11,2) \end{bmatrix}$.

Group 12 $\begin{bmatrix} 17 & 16 & 6 \\ 9 & 10 & 20 \end{bmatrix}$ (a) $\begin{bmatrix} (10,7) & (11,5) & (4,2) \\ (6,3) & (9,1) & (12,8) \end{bmatrix} \to \begin{bmatrix} (10,3) & (11,9) & (4,2) \\ (6,7) & (5,1) & (12,8) \end{bmatrix} \to$

$$\begin{bmatrix} (10,3) & (11,2) & (4,9) \\ (6,7) & (5,8) & (12,1) \end{bmatrix}.$$ (b) $$\begin{bmatrix} (12,5) & (10,6) & (4,2) \\ (8,1) & (7,3) & (11,9) \end{bmatrix} \to \begin{bmatrix} (8,5) & (7,6) & (4,9) \\ (12,1) & (10,3) & (11,2) \end{bmatrix}.$$

(c) $$\begin{bmatrix} (10,7) & (12,4) & (5,1) \\ (6,3) & (8,2) & (11,9) \end{bmatrix} \to \begin{bmatrix} (10,3) & (12,8) & (5,1) \\ (6,7) & (4,2) & (11,9) \end{bmatrix} \to \begin{bmatrix} (10,3) & (12,1) & (5,8) \\ (6,7) & (4,9) & (11,2) \end{bmatrix}.$$

Group 13 $\begin{bmatrix} 17 & 15 & 7 \\ 9 & 11 & 19 \end{bmatrix}$ (a) $$\begin{bmatrix} (11,6) & (10,5) & (4,3) \\ (8,1) & (9,2) & (12,7) \end{bmatrix} \to \begin{bmatrix} (8,6) & (9,5) & (4,7) \\ (11,1) & (10,2) & (12,3) \end{bmatrix} \to$$

$$\begin{bmatrix} (8,6) & (9,4) & (5,7) \\ (11,1) & (10,3) & (12,2) \end{bmatrix} \to \begin{bmatrix} (8,5) & (9,4) & (6,7) \\ (11,2) & (10,3) & (12,1) \end{bmatrix}.$$ (b) $$\begin{bmatrix} (9,8) & (11,4) & (5,2) \\ (6,3) & (10,1) & (12,7) \end{bmatrix} \to$$

$$\begin{bmatrix} (11,8) & (9,4) & (5,2) \\ (6,1) & (10,3) & (12,7) \end{bmatrix} \to \begin{bmatrix} (11,2) & (9,4) & (5,8) \\ (6,7) & (10,3) & (12,1) \end{bmatrix}.$$ (c) $$\begin{bmatrix} (12,5) & (9,6) & (4,3) \\ (7,2) & (10,1) & (11,8) \end{bmatrix} \to$$

$$\begin{bmatrix} (7,5) & (10,6) & (8,3) \\ (12,2) & (9,1) & (11,4) \end{bmatrix} \begin{bmatrix} (7,3) & (10,6) & (8,5) \\ (12,4) & (9,1) & (11,2) \end{bmatrix} \to \begin{bmatrix} (7,6) & (10,3) & (8,5) \\ (12,1) & (9,4) & (11,2) \end{bmatrix}.$$ (d)

$$\begin{bmatrix} (10,7) & (12,3) & (6,1) \\ (5,4) & (9,2) & (11,8) \end{bmatrix} \to \begin{bmatrix} (10,7) & (12,1) & (6,3) \\ (5,4) & (11,2) & (9,8) \end{bmatrix} \to \begin{bmatrix} (10,3) & (12,1) & (6,7) \\ (5,8) & (11,2) & (9,4) \end{bmatrix}.$$ (e)

$$\begin{bmatrix} (12,5) & (11,4) & (6,1) \\ (7,2) & (8,3) & (10,9) \end{bmatrix} \to \begin{bmatrix} (7,5) & (8,4) & (6,9) \\ (12,2) & (11,3) & (10,1) \end{bmatrix} \to \begin{bmatrix} (7,6) & (8,4) & (5,9) \\ (12,1) & (11,3) & (10,2) \end{bmatrix} \to$$

$$\begin{bmatrix} (7,6) & (8,5) & (4,9) \\ (12,1) & (11,2) & (10,3) \end{bmatrix}.$$

(f) $$\begin{bmatrix} (11,6) & (12,3) & (5,2) \\ (8,1) & (7,4) & (10,9) \end{bmatrix} \to \begin{bmatrix} (8,6) & (7,3) & (5,10) \\ (11,1) & (12,4) & (2,9) \end{bmatrix} \begin{bmatrix} (8,5) & (7,3) & (6,10) \\ (11,2) & (12,4) & (1,9) \end{bmatrix} \to$$

$$\begin{bmatrix} (8,5) & (7,6) & (3,10) \\ (11,2) & (12,1) & (4,9) \end{bmatrix}.$$

Group 14 $\begin{bmatrix} 17 & 14 & 8 \\ 9 & 12 & 18 \end{bmatrix}$ (a) $$\begin{bmatrix} (9,8) & (11,3) & (7,1) \\ (5,4) & (10,2) & (12,6) \end{bmatrix} \to \begin{bmatrix} (5,8) & (10,3) & (7,6) \\ (9,4) & (11,2) & (12,1) \end{bmatrix}.$$

(b) $$\begin{bmatrix} (12,5) & (10,4) & (6,2) \\ (8,1) & (9,3) & (11,7) \end{bmatrix} \to \begin{bmatrix} (8,5) & (10,3) & (11,2) \\ (12,1) & (9,4) & (6,7) \end{bmatrix}.$$ (c) $$\begin{bmatrix} (11,6) & (12,2) & (7,1) \\ (5,4) & (9,3) & (10,8) \end{bmatrix} \to$$

$$\begin{bmatrix} (11,2) & (12,6) & (7,1) \\ (9,4) & (5,3) & (10,8) \end{bmatrix} \to \begin{bmatrix} (11,2) & (12,1) & (7,6) \\ (9,4) & (5,8) & (10,3) \end{bmatrix}. \text{(d)} \begin{bmatrix} (9,8) & (10,4) & (5,3) \\ (7,2) & (11,1) & (12,6) \end{bmatrix} \to$$

$$\begin{bmatrix} (9,4) & (10,8) & (5,3) \\ (11,2) & (7,1) & (12,6) \end{bmatrix} \to \begin{bmatrix} (9,4) & (10,3) & (5,8) \\ (11,2) & (7,6) & (12,1) \end{bmatrix}.$$

Group 15 $\begin{bmatrix} 17 & 13 & 9 \\ 9 & 13 & 17 \end{bmatrix}$ (a) $\begin{bmatrix} (12,5) & (10,3) & (8,1) \\ (7,2) & (9,4) & (11,6) \end{bmatrix} \to \begin{bmatrix} (12,1) & (10,3) & (8,5) \\ (7,6) & (9,4) & (11,2) \end{bmatrix}$.

(b) $\begin{bmatrix} (12,5) & (11,2) & (6,3) \\ (8,1) & (9,4) & (10,7) \end{bmatrix} \to \begin{bmatrix} (12,1) & (11,2) & (6,7) \\ (8,5) & (9,4) & (10,3) \end{bmatrix}$. (c) $\begin{bmatrix} (9,8) & (12,1) & (7,2) \\ (5,4) & (10,3) & (11,6) \end{bmatrix} \to$

$\begin{bmatrix} (9,4) & (12,1) & (7,6) \\ (5,8) & (10,3) & (11,2) \end{bmatrix}$. (d) $\begin{bmatrix} (10,7) & (11,2) & (5,4) \\ (6,3) & (12,1) & (9,8) \end{bmatrix} \to \begin{bmatrix} (10,3) & (11,2) & (5,8) \\ (6,7) & (12,1) & (9,4) \end{bmatrix}$.

Group 16 $\begin{bmatrix} 18 & 18 & 3 \\ 8 & 8 & 23 \end{bmatrix}$ No solutions exist

Group 17 $\begin{bmatrix} 18 & 17 & 4 \\ 8 & 9 & 22 \end{bmatrix}$ (a) $\begin{bmatrix} (11,7) & (9,8) & (3,1) \\ (6,2) & (5,4) & (12,10) \end{bmatrix} \to \begin{bmatrix} (6,7) & (5,8) & (12,1) \\ (11,2) & (9,4) & (3,10) \end{bmatrix}$.

Group 18 $\begin{bmatrix} 18 & 16 & 5 \\ 8 & 10 & 21 \end{bmatrix}$ (a) $\begin{bmatrix} (11,7) & (10,6) & (4,1) \\ (5,3) & (8,2) & (12,9) \end{bmatrix} \to \begin{bmatrix} (11,7) & (2,6) & (12,1) \\ (5,3) & (8,10) & (4,9) \end{bmatrix} \to$

$\begin{bmatrix} (11,2) & (7,6) & (12,1) \\ (5,8) & (3,10) & (4,9) \end{bmatrix}$. (b) $\begin{bmatrix} (10,8) & (11,5) & (4,1) \\ (6,2) & (7,3) & (12,9) \end{bmatrix} \to \begin{bmatrix} (10,8) & (3,5) & (4,9) \\ (6,2) & (7,11) & (12,1) \end{bmatrix} \to$

$\begin{bmatrix} (10,3) & (8,5) & (4,9) \\ (6,7) & (2,11) & (12,1) \end{bmatrix}$. (c) $\begin{bmatrix} (10,8) & (11,5) & (3,2) \\ (7,1) & (6,4) & (12,9) \end{bmatrix} \to \begin{bmatrix} (5,8) & (11,10) & (3,2) \\ (7,6) & (1,4) & (12,9) \end{bmatrix} \to$

$\begin{bmatrix} (5,8) & (11,2) & (3,10) \\ (7,6) & (9,4) & (12,1) \end{bmatrix}$. (d) $\begin{bmatrix} (12,6) & (9,7) & (4,1) \\ (5,3) & (8,2) & (11,10) \end{bmatrix} \to \begin{bmatrix} (12,1) & (9,7) & (4,6) \\ (10,3) & (8,2) & (11,5) \end{bmatrix} \to$

$\left[\begin{array}{ccc} (12,1) & (9,4) & (7,6) \\ (10,3) & (8,5) & (11,2) \end{array}\right]$.

Group 19 $\left[\begin{array}{ccc} 18 & 15 & 6 \\ 8 & 11 & 20 \end{array}\right]$ (a) $\left[\begin{array}{ccc} (11,7) & (9,6) & (4,2) \\ (5,3) & (10,1) & (12,8) \end{array}\right] \to \left[\begin{array}{ccc} (11,2) & (9,6) & (4,7) \\ (5,8) & (10,1) & (12,3) \end{array}\right] \to$

$\left[\begin{array}{ccc} (11,2) & (9,4) & (6,7) \\ (5,8) & (10,3) & (12,1) \end{array}\right]$. (b) $\left[\begin{array}{ccc} (12,6) & (10,5) & (4,2) \\ (7,1) & (8,3) & (11,9) \end{array}\right] \to \left[\begin{array}{ccc} (12,1) & (8,5) & (11,2) \\ (7,6) & (10,3) & (4,9) \end{array}\right]$.

(c) $\left[\begin{array}{ccc} (12,6) & (8,7) & (4,2) \\ (5,3) & (10,1) & (11,9) \end{array}\right] \to \left[\begin{array}{ccc} (12,8) & (6,7) & (4,2) \\ (5,1) & (10,3) & (11,9) \end{array}\right] \to \left[\begin{array}{ccc} (12,1) & (6,7) & (4,9) \\ (5,8) & (10,3) & (11,2) \end{array}\right]$.

(d) $\left[\begin{array}{ccc} (10,8) & (12,3) & (4,2) \\ (7,1) & (6,5) & (11,9) \end{array}\right] \to \left[\begin{array}{ccc} (10,3) & (12,8) & (4,2) \\ (7,6) & (1,5) & (11,9) \end{array}\right] \to \left[\begin{array}{ccc} (10,3) & (12,1) & (11,2) \\ (7,6) & (8,5) & (4,9) \end{array}\right]$.

(e) $\left[\begin{array}{ccc} (10,8) & (12,3) & (5,1) \\ (6,2) & (7,4) & (11,9) \end{array}\right] \to \left[\begin{array}{ccc} (10,3) & (12,8) & (5,1) \\ (6,7) & (2,4) & (11,9) \end{array}\right] \to \left[\begin{array}{ccc} (10,3) & (12,1) & (5,8) \\ (6,7) & (9,4) & (11,2) \end{array}\right]$.

Group 20 $\left[\begin{array}{ccc} 18 & 14 & 7 \\ 8 & 12 & 19 \end{array}\right]$ (a) $\left[\begin{array}{ccc} (12,6) & (10,4) & (5,2) \\ (7,1) & (9,3) & (11,8) \end{array}\right] \to \left[\begin{array}{ccc} (7,6) & (9,4) & (5,8) \\ (12,1) & (10,3) & (11,2) \end{array}\right]$.

(b) $\left[\begin{array}{ccc} (12,6) & (9,5) & (4,3) \\ (7,1) & (10,2) & (11,8) \end{array}\right] \to \left[\begin{array}{ccc} (12,1) & (9,10) & (4,3) \\ (7,6) & (5,2) & (11,8) \end{array}\right] \to \left[\begin{array}{ccc} (12,1) & (9,4) & (10,3) \\ (7,6) & (5,8) & (11,2) \end{array}\right]$.

(c) $\left[\begin{array}{ccc} (12,6) & (11,3) & (5,2) \\ (7,1) & (8,4) & (10,9) \end{array}\right] \to \left[\begin{array}{ccc} (12,1) & (11,8) & (5,2) \\ (7,6) & (3,4) & (10,9) \end{array}\right] \to \left[\begin{array}{ccc} (12,1) & (11,2) & (5,8) \\ (7,6) & (3,10) & (4,9) \end{array}\right]$.

(d) $\left[\begin{array}{ccc} (11,7) & (12,2) & (6,1) \\ (5,3) & (8,4) & (10,9) \end{array}\right] \to \left[\begin{array}{ccc} (11,2) & (12,7) & (6,1) \\ (5,8) & (3,4) & (10,9) \end{array}\right] \to \left[\begin{array}{ccc} (11,2) & (12,1) & (6,7) \\ (5,8) & (3,10) & (4,9) \end{array}\right]$.

(e) $\left[\begin{array}{ccc} (10,8) & (9,5) & (4,3) \\ (6,2) & (11,1) & (12,7) \end{array}\right] \to \left[\begin{array}{ccc} (10,8) & (3,5) & (4,9) \\ (6,2) & (11,7) & (12,1) \end{array}\right] \to \left[\begin{array}{ccc} (10,3) & (8,5) & (4,9) \\ (6,7) & (11,2) & (12,1) \end{array}\right]$.

Group 21 $\left[\begin{array}{ccc} 18 & 13 & 8 \\ 8 & 13 & 18 \end{array}\right]$ (a) $\left[\begin{array}{ccc} (11,7) & (12,1) & (6,2) \\ (5,3) & (9,4) & (10,8) \end{array}\right] \to \left[\begin{array}{ccc} (11,2) & (12,1) & (6,7) \\ (5,8) & (9,4) & (10,3) \end{array}\right]$.

(b) $\begin{bmatrix} (12,6) & (11,2) & (7,1) \\ (5,3) & (9,4) & (10,8) \end{bmatrix} \to \begin{bmatrix} (12,1) & (11,2) & (7,6) \\ (5,8) & (9,4) & (10,3) \end{bmatrix}$.

Group 22, 23, 24 $\begin{bmatrix} 19 & 19 & 1 \\ 7 & 7 & 25 \end{bmatrix}, \begin{bmatrix} 19 & 18 & 2 \\ 7 & 8 & 24 \end{bmatrix}, \begin{bmatrix} 19 & 17 & 3 \\ 7 & 9 & 23 \end{bmatrix}$ No solution exist.

Group 25 $\begin{bmatrix} 19 & 16 & 4 \\ 7 & 10 & 22 \end{bmatrix}$ (a) $\begin{bmatrix} (11,8) & (9,7) & (3,1) \\ (5,2) & (6,4) & (12,10) \end{bmatrix} \to \begin{bmatrix} (5,8) & (6,7) & (3,10) \\ (11,2) & (9,4) & (12,1) \end{bmatrix}$.

Group 26 $\begin{bmatrix} 19 & 15 & 5 \\ 7 & 11 & 21 \end{bmatrix}$ (a) $\begin{bmatrix} (12,7) & (9,6) & (4,1) \\ (5,2) & (8,3) & (11,10) \end{bmatrix} \to \begin{bmatrix} (12,1) & (9,6) & (4,7) \\ (11,2) & (8,3) & (5,10) \end{bmatrix} \to$

$\begin{bmatrix} (12,1) & (9,4) & (6,7) \\ (11,2) & (8,5) & (3,10) \end{bmatrix}$. (b) $\begin{bmatrix} (11,8) & (10,5) & (3,2) \\ (6,1) & (7,4) & (12,9) \end{bmatrix} \to \begin{bmatrix} (11,2) & (10,5) & (3,8) \\ (12,1) & (7,4) & (6,9) \end{bmatrix} \to$

$\begin{bmatrix} (11,2) & (10,3) & (5,8) \\ (12,1) & (7,6) & (4,9) \end{bmatrix}$.

Group 27 $\begin{bmatrix} 19 & 14 & 6 \\ 7 & 12 & 20 \end{bmatrix}$ (a) $\begin{bmatrix} (10,9) & (11,3) & (4,2) \\ (6,1) & (7,5) & (12,8) \end{bmatrix} \to \begin{bmatrix} (4,9) & (11,3) & (10,2) \\ (12,1) & (7,5) & (6,8) \end{bmatrix} \to$

$\begin{bmatrix} (4,9) & (11,2) & (10,3) \\ (12,1) & (8,5) & (6,7) \end{bmatrix}$. (b) $\begin{bmatrix} (12,7) & (8,6) & (5,1) \\ (4,3) & (10,2) & (11,9) \end{bmatrix} \to \begin{bmatrix} (12,1) & (8,6) & (5,7) \\ (4,9) & (10,2) & (11,3) \end{bmatrix} \to$

$\begin{bmatrix} (12,1) & (8,5) & (6,7) \\ (4,9) & (11,2) & (10,3) \end{bmatrix}$.

Group 28 $\begin{bmatrix} 19 & 13 & 7 \\ 7 & 13 & 19 \end{bmatrix}$ (a) $\begin{bmatrix} (12,7) & (9,4) & (5,2) \\ (6,1) & (10,3) & (11,8) \end{bmatrix} \to \begin{bmatrix} (12,1) & (9,4) & (5,8) \\ (6,7) & (8,5) & (11,2) \end{bmatrix}$.

(b) $\begin{bmatrix} (10,9) & (11,2) & (4,3) \\ (6,1) & (8,5) & (12,7) \end{bmatrix} \to \begin{bmatrix} (10,3) & (11,2) & (4,9) \\ (6,7) & (8,5) & (12,1) \end{bmatrix}$. (c) $\begin{bmatrix} (11,8) & (12,1) & (5,2) \\ (4,3) & (7,6) & (10,9) \end{bmatrix} \to$

$\begin{bmatrix} (11,2) & (12,1) & (5,8) \\ (4,9) & (7,6) & (10,3) \end{bmatrix}$.

Groups 29, 30, 31 $\begin{bmatrix} 20 & 18 & 1 \\ 6 & 8 & 25 \end{bmatrix}, \begin{bmatrix} 20 & 17 & 2 \\ 6 & 9 & 24 \end{bmatrix}, \begin{bmatrix} 20 & 16 & 3 \\ 6 & 10 & 23 \end{bmatrix}$ No solution exist.

Group 32 $\begin{bmatrix} 20 & 15 & 4 \\ 6 & 11 & 22 \end{bmatrix}$ (a) $\begin{bmatrix} (11,9) & (8,7) & (3,1) \\ (4,2) & (6,5) & (12,10) \end{bmatrix} \rightarrow \begin{bmatrix} (4,9) & (6,7) & (3,10) \\ (11,2) & (8,5) & (12,1) \end{bmatrix}$.

Group 33 $\begin{bmatrix} 20 & 14 & 5 \\ 6 & 12 & 21 \end{bmatrix}$ No solution exist.

Group 34 $\begin{bmatrix} 20 & 13 & 6 \\ 6 & 13 & 20 \end{bmatrix}$ (a) $\begin{bmatrix} (12,8) & (10,3) & (4,2) \\ (5,1) & (7,6) & (11,9) \end{bmatrix} \rightarrow \begin{bmatrix} (12,1) & (10,3) & (4,9) \\ (5,8) & (7,6) & (11,2) \end{bmatrix}$.

Group 35-38 $\begin{bmatrix} 21 & 17 & 1 \\ 5 & 9 & 25 \end{bmatrix}, \begin{bmatrix} 21 & 16 & 2 \\ 5 & 10 & 24 \end{bmatrix}, \begin{bmatrix} 21 & 15 & 3 \\ 5 & 11 & 23 \end{bmatrix}, \begin{bmatrix} 21 & 14 & 4 \\ 5 & 12 & 22 \end{bmatrix}$ No solution exist.

Group 39 $\begin{bmatrix} 21 & 13 & 5 \\ 5 & 13 & 21 \end{bmatrix}$ (a) $\begin{bmatrix} (12,9) & (8,5) & (4,1) \\ (3,2) & (7,6) & (11,10) \end{bmatrix} \rightarrow \begin{bmatrix} (12,1) & (8,5) & (4,9) \\ (3,10) & (7,6) & (11,2) \end{bmatrix}$.

Group 40-46 $\begin{bmatrix} 22 & 16 & 1 \\ 4 & 10 & 25 \end{bmatrix}, \begin{bmatrix} 22 & 15 & 2 \\ 4 & 11 & 24 \end{bmatrix}, \begin{bmatrix} 22 & 14 & 3 \\ 4 & 12 & 23 \end{bmatrix}, \begin{bmatrix} 22 & 13 & 4 \\ 4 & 13 & 22 \end{bmatrix},$

$\begin{bmatrix} 23 & 15 & 1 \\ 3 & 11 & 25 \end{bmatrix}, \begin{bmatrix} 23 & 14 & 2 \\ 3 & 12 & 24 \end{bmatrix}, \begin{bmatrix} 23 & 13 & 3 \\ 3 & 13 & 23 \end{bmatrix}$. No solution exist.

8 An Alternate Method for Computing All Solutions to the Magic 2×3 Checkerboard

Let T denote the collection of four-element subsets of

$$u = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

whose sum is 26.

The members of T can be listed in lexicographic order: 1, 2, 11, 12; 1, 3, 10, 12; 1, 4, 9, 12; 1, 4, 10, 11; 1, 5, 8, 12; 1, 5, 9, 11; 1, 6, 7, 12; 1, 6, 8, 11; 1, 6, 9, 10; 1, 7, 8, 10; 2, 3, 9, 12; 2, 3, 10, 11; 2, 4, 8, 12; 2, 4, 9, 11; 2, 5, 7, 12; 2, 5, 8, 11; 2, 5, 9, 10; 2, 6, 7, 11; 2, 6, 8, 10; 2, 7, 8, 9; 3, 4, 7, 12; 3, 4, 8, 11; 3, 4, 9, 10; 3, 5, 6, 12; 3, 5, 7, 11; 3, 5, 8, 10; 3, 6, 7, 10; 3, 6, 8, 9; 4, 5, 6, 11; 4, 5, 7, 10; 4, 5, 8, 9; 4, 6, 7, 9; 5, 6, 7, 8 for a total of 33 sets.

We call a triplet α, β, γ of elements of T compatible provided they are pairwise disjoint. There are 32 compatible triplets and each of the 33 members of T belongs to either 2, 3 or 4 of these triplets.

After we list the 32 compatible triplets, they can be used to systematically compute all of the solutions to the 2×3 magic checkerboard puzzle by letting each compatible triplet represent the four numbers in the 3 columns. There are between 2 and 8 solutions (up to the trivial equivalency classes defined in Section 2) to the 2×3 magic checkerboard puzzle for each compatible triplet α, β, γ.

9 A Discussion

The underlying structure of many puzzles is that the solver must follow a connected undirected graph from one vertex i (called the initial position) to a second vertex j (called the final position). Of course, this underlying graph is almost never given. This class of puzzles can become arbitrarily difficult. The puzzles in this paper are probably much too difficult to be commercially attractive, but they seem similar to a collection of Binary Arts Puzzles called Rush Hour, which appeared on the market around 1995.

References

[1] This problem, called the Magic Double Hexagon, was posed by Nadejda Dykevich in the New York Numberplay Blog by Gary Antonick during the week of July 17, 2013.