A Theory of Linear Fractional Transformations of Rational Functions

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1 Abstract

If $g, \overline{g}$ are complex rational functions, we say that $g \sim \overline{g}$ if $g = \left(\frac{ax+b}{cx+d}\right)^{-1} \circ \overline{g} \circ \left(\frac{ax+b}{cx+d}\right)$, where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$. For practical purposes, the general problem of finding a collection of rational invariants that are sufficient to partition $\sim$ into equivalency classes may be intractable for arbitrary degree rational functions.

In this paper, we first outline a simple and naive meta-method for finding weak rational invariants when g and $\overline{g}$ satisfy $g \sim \overline{g}$. These 'weak' invariants can be combined to create 'strong' invariants. This meta-method makes the invariants seem almost self-evident. We now know that there is a very large number of these weak rational invariants which we divide into two levels.

We apply this meta-method by finding three first level invariants that hold for arbitrary degree rational functions. Then we give alternate proofs using the well-known theory of resultants. These proofs are on the same level as the theorems themselves. In some special cases such as when $\frac{ax+b}{cx+d} = ax + b$, a linear function, our methods yield a large number of first level invariants which can also be extended to an infinite number. Also, by giving the reader one single axiom, our methods can be completely understood by a naive person. This is in sharp contract to the classical theory of invariants (see [2]) which is very specialized. At the end, we state necessary and sufficient conditions so that $\frac{Ax^2+Bx+C}{Hx^2+DX+E} \sim \frac{\overline{A}x^2+\overline{B}x+\overline{C}}{\overline{H}x^2+\overline{D}x+\overline{E}}$.

The applications in this paper deal exclusively with first level invariants. However, we have computed and independently verified nine second level invariants for rational quadratics. This gives a total of 12 weak invariants for rational quadratics, and each of these invariants has a different meaning. These 12 invariants cannot possibly be independent. As an obvious extension of our methods, we also state an unproven algorithm which computes from scratch

a large number of weak invariants for rational functions of any degree. This algorithm computes 38 'weak' invariants for rational quadratics. Some of these 38 weak invariants are identical. Also, they include most but not all of the above first and second level invariants. Thus, our methods can compute about 30 interrelated weak invariants for rational quadratics. These have all been independently verified and the relations classified. Therefore, in some ways our elementary theory is as potent as the specialized classical theory of invariants.

2 Introductory Concepts

A rational function g of degree n is a function of the form $g = \sum_{i=0}^{n} A_i x^i / \sum_{i=0}^{n} B_i x^i$ where $(A_n, B_n) \neq (0,0)$, at least one $A_i \neq 0$, at least one $B_i \neq 0$ and $\sum_{i=0}^{n} A_i x^i$ and $\sum_{i=0}^{n} B_i x^i$ have no roots in common.

Also, $g \sim \overline{g}$ if $g = \left(\frac{ax+b}{cx+d}\right)^{-1} \circ \overline{g} \circ \left(\frac{ax+b}{cx+d}\right)$ where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$. If g is a rational function of degree n and $f = \frac{ax+b}{cx+d}, \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, then Lemmas 6 and 8 in this paper will prove that both $g \circ f$ and $f \circ g$ are of degree n. However, in order for our 'weak' invariants to make sense, we will now define things in a more rigid way.

Definition 1 *Let $P(x) = \sum_{i=0}^{n} A_i x^i, A_n \neq 0$, be an nth degree polynomial and suppose $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0, a, b, c, d \in \boldsymbol{C}$. In isolated degenerate cases where $A_n = 0$, we will still consider $P(x)$ to be of degree n.*

Define $P\left(\left(\frac{ax+b}{cx+d}\right)\right) = P \circ \left(\left(\frac{ax+b}{cx+d}\right)\right) = (cx+d)^n \cdot P\left(\frac{ax+b}{cx+d}\right) = \sum_{i=0}^{n} A_i (ax+b)^i (cx+d)^{n-i}$.

Note 1. If $t \neq 0$, then $P\left(\left(\frac{atx+bt}{ctx+dt}\right)\right) = t^n \cdot P\left(\left(\frac{ax+b}{cx+d}\right)\right)$. However, we usually consider a, b, c, d to be fixed, and we do not deal with $\frac{ax+b}{cx+d} = \frac{atx+bt}{ctx+dt}, t \neq 0$. In order for our 'weak' invariants to make sense, we likewise do not consider $\frac{P(x)}{Q(x)} = \frac{tP(x)}{tQ(x)}, \frac{\overline{P}(x)}{\overline{Q}(x)} = \frac{t\overline{P}(x)}{t\overline{Q}(x)}, t \neq 0$.

When we combine our 'weak' invariants to define 'strong' invariants, then in the normal way we can allow $\frac{ax+b}{cx+d} = \frac{atx+bt}{ctx+dt}, \frac{P(x)}{Q(x)} = \frac{tP(x)}{tQ(x)}, \frac{\overline{P}(x)}{\overline{Q}(x)} = \frac{t\overline{P}(x)}{t\overline{Q}(x)}, t \neq 0$.

Lemma 1 *Let $P(x) = \sum_{i=0}^{n} A_i x^i, A_n \neq 0, Q(x) = \sum_{i=0}^{n} B_i x^i, B_n \neq 0$, and suppose that $P(x), Q(x)$ have no roots in common. Also, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$. In isolated degenerate cases where one of $A_n = 0$ or $B_n = 0$, we will still compute this the same way.*

Then $\left(\frac{ax+b}{cx+d}\right)^{-1}\circ\frac{P(x)}{Q(x)}\circ\left(\frac{ax+b}{cx+d}\right)=\frac{dx-b}{a-cx}\circ\frac{P\left(\left(\frac{ax+b}{cx+d}\right)\right)}{Q\left(\left(\frac{ax+b}{cx+d}\right)\right)}=\frac{\overline{P}(x)}{\overline{Q}(x)}$ *where* $\overline{P}(x),\overline{Q}(x)$ *are defined as follows.*

$\overline{P}(x)=dP\left(\left(\frac{ax+b}{cx+d}\right)\right)-bQ\left(\left(\frac{ax+b}{cx+d}\right)\right)$

$\overline{Q}(x)=-cP\left(\left(\frac{ax+b}{cx+d}\right)\right)+aQ\left(\left(\frac{ax+b}{cx+d}\right)\right).$

Of course, $\overline{P}(x),\overline{Q}(x)$ *have no roots in common.*

Definition 2 *Using the hypothesis of Lemma 1 and computing isolated degenerate cases where one of* $A_n=0$ *or* $B_n=0$ *in the same way, we say that* $\frac{P(x)}{Q(x)}\sim\frac{\overline{P}(x)}{\overline{Q}(x)}$ *if there exist* $a,b,c,d\in\boldsymbol{C},\begin{vmatrix} a & b\\ c & d\end{vmatrix}\neq 0$*, such that we can transform* $\frac{P(x)}{Q(x)}\to\frac{\overline{P}(x)}{\overline{Q}(x)}$ *by the transformation of Lemma 1.*

Note 2. First, note that if $t\neq 0$ then Lemma 1 computes $\left(\frac{atx+bt}{ctx+dt}\right)^{-1}\circ\frac{P(x)}{Q(x)}\circ\left(\frac{atx+bt}{ctx+dt}\right)=\frac{t^{n+1}\cdot\overline{P}(x)}{t^{n+1}\cdot\overline{Q}(x)}$ where $\overline{P}(x),\overline{Q}(x)$ are defined in Lemma 1. Also, by using $\frac{ax+b}{cx+d}=\frac{tx+0}{0x+t},t\neq 0$, and then adjusting t we easily see that for all $c\neq 0$, $\frac{P(x)}{Q(x)}\sim\frac{cP(x)}{cQ(x)}$. Thus, even though Lemma 1 rigidly defines $\overline{P}(x),\overline{Q}(x)$ we can still prove that $\sim$ of Definition 2 is an equivalence relation.

We also emphasize again that we usually consider a,b,c,d to be fixed and for our 'weak' invariants to make sense, we must always define $\overline{P}(x),\overline{Q}(x)$ exactly as in Lemma 1.

Also, in Lemma 1 we note that if $\frac{ax+b}{cx+d}=\frac{ax+0}{0x+1}$ then $\overline{P}(x)=P(ax),\overline{Q}(x)=aQ(ax)$.

Also, we need to point out that we will consider all polynomials such as $P(x)=\sum_{i=0}^{n}A_ix^i,Q(x)=\sum_{i=0}^{n}B_ix^i$ to be of the degree that we think they should be. Thus, $P(x)$ and $Q(x)$ are of degree n. We will also consider $P\left(\left(\frac{ax+b}{cx+d}\right)\right),Q\left(\left(\frac{ax+b}{cx+d}\right)\right),\overline{P}(x),\overline{Q}(x)$ to be polynomials of degree n.We can 'patch up' degenerate cases where come of the leading coefficients are zeros by using continuity arguments.

In general, polynomials are so well-behaved that all types of isolated degenerate cases are 'swallowed up' by continuity.

However, the easiest way to deal with the isolated degenerate cases in this paper is just to consider all calculations to be symbolic algebra. Then these degenerate cases do not arise.

Lemma 2 *Using* $\overline{P}(x),\overline{Q}(x)$ *from Lemma 1, denote* $\overline{P}(x)=\sum_{i=0}^{n}\overline{A}_i(a,b,c,d)\,x^i$ *and* $\overline{Q}(x)=\sum_{i=0}^{n}\overline{B}_i(a,b,c,d)\,x^i$ *where* $\overline{A}_i(a,b,c,d),\overline{B}_i(a,b,c,d)$ *are polynomials. Then for each* $0\leq r\leq n,\overline{A}_r(a,b,c,d)=c^r\cdot d^{n+1-r}\cdot\overline{A}_r\left(\frac{a}{c},\frac{b}{d}\right),\overline{B}_r(a,b,c,d)=c^{1+r}\cdot d^{n-r}\cdot\overline{B}_r\left(\frac{a}{c},\frac{b}{d}\right)$ *where* $\overline{A}_r\left(\frac{a}{c},\frac{b}{d}\right),\overline{B}_r\left(\frac{a}{c},\frac{b}{d}\right)$ *are polynomials in the two variables* $\frac{a}{c},\frac{b}{d}$.

Proof. $\overline{P}(x)=d\sum_{i=0}^{n}A_i(ax+b)^i(cx+d)^{n-i}-b\sum_{i=0}^{n}B_i(ax+b)^i(cx+d)^{n-i}.$

Let $0 \le r \le n$ be arbitrary but fixed.

In $\overline{P}(x)$, first consider an arbitrary term $dA_t (ax+b)^t (cx+d)^{n-t}$ where $0 \le t \le n$ and t is arbitrary but fixed.

Also, let $i + j = r$ with i, j arbitrary but subject to $0 \le i \le t, 0 \le j \le n - t$.

Now an arbitrary ith term in $(ax+b)^t$ is of the form $a^i b^{t-i} x^i$ and an arbitrary jth term in $(cx+d)^{n-t}$ is of the form $c^j d^{n-t-j} x^j$.

Now

$$\begin{aligned} da^i b^{t-i} x^i \cdot c^j d^{n-t-j} x^j &= a^i b^{t-i} c^j d^{1+n-t-j} x^{i+j} \\ &= a^i b^{t-i} c^{r-i} d^{1+n-t+i-r} x^r \\ &= c^r \cdot d^{1+n-r} \cdot \left(\frac{a}{c}\right)^i \left(\frac{b}{d}\right)^{t-i} x^r. \end{aligned}$$

Also,

$$\begin{aligned} & b \sum_{i=0}^{n} B_i (ax+b)^i (cx+d)^{n-i} \\ = \; & d \sum_{i=0}^{n} \left(\frac{b}{d}\right) B_i (ax+b)^i (cx+d)^{n-i} \end{aligned}$$

and the same proof holds. Also, $\overline{Q}(x) = \left(\frac{c}{d}\right) \left[-dP\left(\left(\frac{ax+b}{cd+d}\right)\right) + \left(\frac{a}{c}\right) dQ\left(\left(\frac{ax+b}{cd+d}\right)\right)\right]$ and the same proof again holds. ■

3 An Important Property of Polynomials

Lemma 3 *Suppose $P(x, y, z, v)$ is a polynomial having the property that for all $x, y, z, v \in$ $\boldsymbol{C}$, if $\begin{vmatrix} x & y \\ z & v \end{vmatrix} \ne 0$ then $P(x, y, z, v) \ne 0$. Then $P(x, y, z, v) = \overline{c} \cdot (xv - yz)^n$ where $\overline{c} \ne 0$ is a constant.*

Proof. Of course, $P(x, y, z, v) \ne 0$, the zero polynomial. If $P(x, y, z, v)$ is not constant, by symmetry we may suppose $P(x, y, z, v) = \sum_{i=0}^{n} P_i(y, z, v) x^i$ where $n \ge 1$, each $P_i(y, z, v)$ is a polynomial in y, z, v and $P_n(y, z, v) \ne 0$, the zero polynomial. Now $v \cdot P_n(y, z, v) \ne 0$, the zero polynomial. Since $v \cdot P_n(y, z, v)$ is continuous and $v \cdot P_n(y, z, v) \ne 0$, there exists an open ball S^* in the 3-dimensional space of y, z, v such that $v \cdot P_n(y, z, v) \ne 0$ for all $(y, z, v) \in S^*$. Since $v \ne 0$ and $P_n(y, z, v) \ne 0$ on S^*, we know that if $(y, z, v) \in S^*, x \in \mathbf{C}$ and $x \ne \frac{yz}{v}$ then by the hypothesis $P(x, y, z, v) \ne 0$.

Therefore, for each $(y, z, v) \in S^*$, we know that all x-roots of $P(x, y, z, v) = 0$ must be $x = \frac{yz}{v}$, and we conclude that for all $(y, z, v) \in S^*, P(x, y, z, v) = P_n(y, z, v) \cdot \left(x - \frac{yz}{v}\right)^n =$

$\frac{P_n(y,z,v)\cdot(xv-yz)^n}{v^n}$. Now $v^n \cdot P(x,y,z,v) = P_n(y,z,v)\cdot(xv-yz)^n$ for all $(y,z,v)\in S^*$ and all $x\in\mathbf{C}$.

Therefore, from algebra and analysis, we know that $v^n \cdot P(x,y,z,v) = P_n(y,z,v)\cdot(xv-yz)^n$ must be true for all $x,y,z,v\in\mathbf{C}$. Therefore, $P(x,y,z,v) = \frac{P_n(y,z,v)\cdot(xv-yz)^n}{v^n}$.

Now $\frac{xv-yz}{v}$ is already reduced to its lowest terms. Therefore, since $\frac{P(y,z,v)(xv-yz)^n}{v^n}$ is a polynomial, we know from a simple analogy of the theory of primitive polynomials (Gauss) that $\frac{P_n(y,z,v)}{v^n} = \overline{P}(y,z,v)$ where $\overline{P}(y,z,v)$ is a polynomial in y,z,v.

Therefore, $P(x,y,z,v) = \overline{P}(y,z,v)\cdot(xv-yz)^n$.

Now if $\overline{P}(y,z,v)$ is a constant, there is nothing to prove. Therefore, suppose $\overline{P}(y,z,v)$ is non-constant. Suppose $\overline{P}(y,z,v) = \overline{P}(y,z)$ where v is absent. Let $\overline{P}(\overline{y},\overline{z}) = 0$. If we define $\overline{x},\overline{v}$ so that $\overline{xv}-\overline{yz}\neq 0$ then $P(\overline{x},\overline{y},\overline{z},\overline{v}) = \overline{P}(\overline{y},\overline{z})(\overline{xv}-\overline{yz})^n = 0$ and $\begin{vmatrix}\overline{x} & \overline{y}\\ \overline{z} & \overline{v}\end{vmatrix}\neq 0$ which is a contradiction.

Therefore, $\overline{P}(y,z,v) = \sum_{i=0}^{m}\overline{P}_i(y,z)\,v^i$ where each $\overline{P}_i(y,z)$ is a polynomial, $m\geq 1$ and $\overline{P}_m(y,z)\neq 0$, the zero polynomial. First, suppose $\overline{P}_0(y,z)=0$, the zero polynomial. Then $\overline{P}(y,z,v)=0$ for $v=0$ and y,z arbitrary.

Therefore, if we choose $\overline{x},\overline{y},\overline{z}\in\mathbf{C},\overline{v}=0$ so that $\overline{xv}-\overline{yz}=-\overline{yz}\neq 0$ then $P(\overline{x},\overline{y},\overline{z},\overline{v}) = \overline{P}(\overline{y},\overline{z},0)(\overline{xv}-\overline{yz})^n = 0$ which is a contradiction.

Therefore, $\overline{P}_0(y,z)\neq 0$. Now since $\overline{P}_0(y,z)\neq 0$ and $\overline{P}_m(y,z)\neq 0$ we know that $\overline{P}_m(y,z)\cdot\overline{P}_0(y,z)\neq 0$, the zero polynomial.

Therefore, there exist $\overline{y},\overline{z}\in\mathbf{C}$ such that $\overline{P}_m(\overline{y},\overline{z})\neq 0$ and $\overline{P}_0(\overline{y},\overline{z})\neq 0$. Therefore, there exist $\overline{y},\overline{z},\overline{v}\in\mathbf{C}$ with $\overline{v}\neq 0$ such that $\overline{P}(\overline{y},\overline{z},\overline{v})=0$. If we now choose $\overline{x}$ such that $\overline{xv}-\overline{yz}\neq 0$, then we have $P(\overline{x},\overline{y},\overline{z},\overline{v}) = \overline{P}(\overline{y},\overline{z},\overline{v})(\overline{xv}-\overline{yz})^n = 0$ which is a contradiction. Therefore, the assumption that $\overline{P}(y,z,v)$ is non-constant is incorrect which completes the proof. ■

4 A Naive Meta-method for Deriving the Invariants

The method in this section allows us to derive invariants for the polynomials $\frac{P(x)}{Q(x)}\rightarrow\frac{\overline{P}(x)}{\overline{Q}(x)}$ that are specified in Lemma 1.

The method is so convincing that one almost wonders whether these invariants must be proved at all. Using the definitions of $P(x), Q(x), \overline{P}(x), \overline{Q}(x)$ in Lemma 1, then from Lemma 2 we can write $\overline{P}(x), \overline{Q}(x)$ as follows.

$$\begin{aligned}
\overline{P}(x) &= \sum_{i=0}^{n} \overline{A}_i x^i = \sum_{i=0}^{n} \overline{A}_i (a,b,c,d)\, x^i \\
&= \sum_{i=0}^{n} c^i d^{n+1-i} \overline{A}_i \left(\frac{a}{c}, \frac{b}{d}\right) x^i, \\
\overline{Q}(x) &= \sum_{i=0}^{n} \overline{B}_i x^i = \sum_{i=0}^{n} \overline{B}_i (a,b,c,d)\, x^i \\
&= \sum_{i=0}^{n} c^{i+1} d^{n-i} \overline{B}_i \left(\frac{a}{c}, \frac{b}{d}\right) x^i.
\end{aligned}$$

Of course, a, b, c, d are variables and the coefficients of $P(x) = \sum_{i=0}^{n} A_i x^i, Q(x) = \sum_{i=0}^{n} B_i x^i$ are considered to be constants. Also, the coefficients of the polynomials $\overline{A}_i (a,b,c,d), \overline{A}_i \left(\frac{a}{c}, \frac{b}{d}\right)$ and $\overline{B}_i (a,b,c,d), \overline{B}_i \left(\frac{a}{c}, \frac{b}{d}\right)$ are given in terms of the A_i's, B_i's.

In order to 'patch up' degeneracies that can arise, we are going to consider $A_1, \cdots, A_n$, $B_1, \cdots, B_n, a, b, c, d$ to be symbolic letters, and we will manipulate by symbolic algebra. Since each $\overline{A}_i \left(\frac{a}{c}, \frac{b}{d}\right), \overline{B}_i \left(\frac{a}{c}, \frac{b}{d}\right)$ has two degrees of freedom, in general (but not necessary always) we might expect that we could choose $\frac{a}{c}, \frac{b}{d}, \frac{a}{c} \neq \frac{b}{d}$, so as to force any two of the $\overline{A}_i \left(\frac{a}{c}, \frac{b}{d}\right)$'s, $\overline{B}_i \left(\frac{a}{c}, \frac{b}{d}\right)$'s to be zero. We require $\frac{a}{c} \neq \frac{b}{d}$ since $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, and we will not consider $c = 0$ or $d = 0$ since we are manipulating symbolically.

For example, we might force $\overline{A}_2 \left(\frac{a}{c}, \frac{b}{d}\right) = 0, \overline{B}_3 \left(\frac{a}{c}, \frac{b}{d}\right) = 0$. Also, in general, if the coefficients of $P(x) = \sum_{i=0}^{n} A_i x^i, Q(x) = \sum_{i=0}^{n} B_i x^i$ are related in such a special way that we could force a certain collection of three of the $\overline{A}_i \left(\frac{a}{c}, \frac{b}{d}\right)$'s, $\overline{B}_i \left(\frac{a}{c}, \frac{b}{d}\right)$'s, to be zero when $\frac{a}{c} \neq \frac{b}{d}$, then this would be exceptional. Since we are manipulating symbolically, it seems almost self evident that this exceptional relation of the A_i's of $P(x)$ and B_i's of $Q(x)$ must remain invariant under the transformation $\sim$ that we are discussing. This is because no matter how we transform $\frac{P(x)}{Q(x)} \rightarrow \frac{\overline{P}^*(x)}{\overline{Q}^*(x)}$, we could still transform $\frac{\overline{P}^*(x)}{\overline{Q}^*(x)} \rightarrow \frac{\overline{P}(x)}{\overline{Q}(x)}$ so that these same three $\overline{A}_i \left(\frac{a}{c}, \frac{b}{d}\right)$'s, $\overline{B}_i \left(\frac{a}{c}, \frac{b}{d}\right)$'s would be zero. This is because the relation $\sim$ of Definition 2 is an equivalence relation. (see Note 2). So all we would have to do is back up $\frac{\overline{P}^*(x)}{\overline{Q}^*(x)} \rightarrow \frac{P(x)}{Q(x)}$, then transform it again $\frac{P(x)}{Q(x)} \rightarrow \frac{\overline{P}(x)}{\overline{Q}(x)}$ and then combine these two transformations to get $\frac{\overline{P}^*(x)}{\overline{Q}^*(x)} \rightarrow \frac{\overline{P}(x)}{\overline{Q}(x)}$. Again see Note 2.

Since we are dealing symbolically, we believe that we have patched up isolated degeneracies that can occur.

Nonetheless, to be on the safe side, we will call the method that soon follows an axiom.

Of course, from the above, it follows that if a certain fixed collection of three of the $\overline{A}_i\left(\frac{a}{c},\frac{b}{d}\right)$'s, $\overline{B}_i\left(\frac{a}{c},\frac{b}{d}\right)$'s cannot be forced to be zero for $\frac{a}{c}\neq\frac{b}{d}$, then this property would also be an invariant under $\sim$.

Let us call $\frac{a}{c}=t$, $\frac{b}{d}=s$. Now, s$\neq t$ since $\begin{vmatrix} a & b \\ c & d \end{vmatrix}\neq 0$ is symbolically, the same as $s\neq t$. Consider three fixed $\overline{A}_i\left(s,t\right)$'s, $\overline{B}_j\left(s,t\right)$'s. Also, as in Section 6 in some exceptional cases we consider two $\overline{A}_i\left(s,t\right)$'s, $\overline{B}_j\left(s,t\right)$'s.

For illustration, consider $\overline{A}_i\left(s,t\right)$,$\overline{A}_j\left(s,t\right)$, and $\overline{B}_k\left(s,t\right)$. As always, the coefficients of $\overline{A}_i\left(s,t\right), \overline{A}_j\left(s,t\right)$ and $B_k(s,t)$ are given in terms of the original coefficients of $P\left(x\right)=\sum_{i=0}^{n}A_ix^i$, $Q\left(x\right)=\sum_{i=0}^{n}B_ix^i$, and these coefficients are polynomials in $A_1,\ldots,A_n,B_1,\ldots,B_n$.

5 General Principle (Axiom)

Suppose there exists a polynomial $F\left(A_1,\cdots,A_n,B_1,\cdots,B_n\right)$, (where $P\left(x\right)=\sum_{i=0}^{n}A_ix^i$, $Q\left(x\right)=\sum_{i=0}^{n}B_ix^i$), that has property (A).

(A). The three simultaneous equations $\overline{A}_i\left(s,t\right)=0$,$\overline{A}_j\left(s,t\right)=0$, $\overline{B}_k\left(s,t\right)=0$ has a solution s,t with $s\neq t$, if and only if $F\left(A_1,\cdots,A_n,B_n,\cdots,B_n\right)=0$.

Let us now suppose that $A_1,\cdots,A_n,B_1,\cdots,B_n$ are arbitrary but fixed subject only to the condition $F\left(A_1,\cdots,A_n,B_1,\cdots,B_n\right)\neq 0$. Of course, this means that the three simultaneous equations $\overline{A}_i\left(s,t\right)=0,\overline{A}_j\left(s,t\right)=0,\overline{B}_k\left(s,t\right)=0$, do not have a solution $s,t,s\neq t$.

As always, suppose $\frac{P(x)}{Q(x)}\rightarrow\frac{\overline{\overline{P}}(x)}{\overline{\overline{Q}}(x)}$ where $\overline{\overline{P}}\left(x\right)=\sum_{i=0}^{n}\overline{\overline{A}}_i\left(\overline{a},\overline{b},\overline{c},\overline{d}\right)x^i$ and $\overline{\overline{Q}}\left(x\right)=\sum_{i=0}^{n}\overline{\overline{B}}_i\left(\overline{a},\overline{b},\overline{c},\overline{d}\right)x^i$, are computed in Lemma 1. (Note that we are now using new variables $\overline{a},\overline{b},\overline{c},\overline{d}$). Using the same reasoning as in Section 4 it follows that

$$\overline{F}(\overline{a},\overline{b},\overline{c},\overline{d})=F\left(\overline{\overline{A_1}}\left(\overline{a},\overline{b},\overline{c},\overline{d}\right),\cdots,\overline{\overline{A}}_n\left(\overline{a},\overline{b},\overline{c},\overline{d}\right),\overline{\overline{B}}_1\left(\overline{a},\overline{b},\overline{c},\overline{d}\right),\cdots,\overline{\overline{B}}_n\left(\overline{a},\overline{b},\overline{c},\overline{d}\right)\right)\neq 0$$

for all $\overline{a},\overline{b},\overline{c},\overline{d}\in\mathbf{C}$ that satisfies $\begin{vmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{vmatrix}\neq 0$. Since $\overline{F}\left(\overline{a},\overline{b},\overline{c},\overline{d}\right)$ is a polynomial in the variables $\overline{a},\overline{b},\overline{c},\overline{d}$, we know from Lemma 3 that $\overline{F}\left(\overline{a},\overline{b},\overline{c},\overline{d}\right)=\overline{C}^*\begin{vmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{vmatrix}^n$, $\overline{C}^*\neq 0$. If $\overline{a}=1,\overline{b}=\overline{c}=0,\overline{d}=1$, then from Lemma 1 (Also see Note 2), $\overline{\overline{P}}\left(x\right)=P\left(x\right),\overline{\overline{Q}}\left(x\right)=Q\left(x\right)$ which implies that each $\overline{\overline{A}}_i\left(1,0,0,1\right)=A_i,\overline{\overline{B}}_i\left(1,0,0,1\right)=B_i$.

Therefore, when $\left(\overline{a}, \overline{b}, \overline{c}, \overline{d}\right) = (1, 0, 0, 1)$, we have $\overline{F}\,(1, 0, 0, 1) = F\,(A_1, \cdots, A_n, B_1, \cdots, B_n) = \overline{C}^* \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}^n = \overline{C}^*$. Therefore,

$$(*) \qquad F\left(\overline{\overline{A}}_1\left(\overline{a}, \overline{b}, \overline{c}, \overline{d}\right), \cdots, \overline{\overline{A}}_n\left(\overline{a}, \overline{b}, \overline{c}, \overline{d}\right), \overline{\overline{B}}_1\left(\overline{a}, \overline{b}, \overline{c}, \overline{d}\right), \cdots, \overline{\overline{B}}_n\left(\overline{a}, \overline{b}, \overline{c}, \overline{d}\right)\right)$$

$$= F\,(A_1, \cdots, A_n, B_1, \cdots, B_n) \cdot \begin{vmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{vmatrix}^n.$$

Of course, $(A_1, \cdots, A_n, B_1, \cdots, B_n)$ are considered fixed.

The exponent n can be computed from the special case $\overline{a} = \overline{a}, \overline{b} = \overline{c} = 0, \overline{d} = 1, \overline{\overline{P}}\,(x) = P\,(\overline{a}x)\,, \overline{\overline{Q}}\,(x) = \overline{a}Q\,(\overline{a}x)$. See Note 2.

Therefore, $\overline{\overline{A}}_i\,(\overline{a}, 0, 0, 1) = \overline{a}^i A_i$, $\overline{\overline{B}}_i\,(\overline{a}, 0, 0, 1) = \overline{a}^{i+1} B_i$, and n can be computed for the fixed $(A_1, \ldots, A_n, B_1, \ldots, B_n)$.

Of course,

$$F\left(\overline{\overline{A}}_1\left(\overline{a}, \overline{b}, \overline{c}, \overline{d}\right), \cdots, \overline{\overline{A}}_n\left(\overline{a}, \overline{b}, \overline{c}, \overline{d}\right), \overline{\overline{B}}_1\left(\overline{a}, \overline{b}, \overline{c}, \overline{d}\right), \cdots, \overline{\overline{B}}_n\left(\overline{a}, \overline{b}, \overline{c}, \overline{d}\right)\right)$$

is a polynomial in the variables $A_1, \ldots, A_n, B_1, \ldots, B_n, \overline{a}, \overline{b}, \overline{c}, \overline{d}$. Also, for a fixed positive integer n, $F(A_1, \ldots, A_n, B_1, \ldots, B_n) \cdot \begin{vmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{vmatrix}^n$ is a polynomial in the variables $A_1, \ldots, A_n$, $B_1, \ldots, B_n, \overline{a}, \overline{b}, \overline{c}, \overline{d}$. It is fairly easy to show that there exists (n, S^*) where n is a fixed positive integer and S^* is a fixed open ball in the Euclidean space of the variables $A_1, \ldots, A_n$, $B_1, \ldots, B_n, \overline{a}, \overline{b}, \overline{c}, \overline{d}$ such that $(*)$ is true on S^*. Therefore, from algebra and analysis, we know that $(*)$ must be true for all $A_1, \ldots, A_n, B_1, \ldots, B_n, \overline{a}, \overline{b}, \overline{c}, \overline{d} \in \mathbf{C}$.

This means that we have computed an invariant for the transformation $\frac{P(x)}{Q(x)} \to \frac{\overline{\overline{P}}(x)}{\overline{\overline{Q}}(x)}$ of Lemma 1 when $\overline{\overline{P}}\,(x)\,, \overline{\overline{Q}}\,(x)$ are computed exactly as in Lemma 1. We call this type of invariant a weak invariant. All algorithms in this paper generate weak invariants.

Definition 3 *If (as in Section 6) the General Principle works for two coefficients* $\left\{\overline{A}_i\,(s,t)\,, \overline{A}_j\,(s,t)\right\}$ *or* $\left\{\overline{A}_i\,(s,t)\,, \overline{B}_j\,(s,t)\right\}$ *or* $\left\{\overline{B}_i\,(s,t)\,, \overline{B}_j\,(s,t)\right\}$*, then we call the invariant a first level invariant. Otherwise, we call the invariant a second level invariant.*

Observation 1 In Section 6 we illustrate a first level invariant that is computed by using the resultant of two polynomials. (See definition 4.)

Also, resultants can be used when one of the three polynomials $\overline{A}_i\,(s,t)\,, \overline{A}_j\,(s,t)\,, \overline{B}_k\,(s,t)$ is linear in one of the two variables s, t.

To rigorously use the General Principle, it is extremely helpful to note that if $s = t$, then for each i, j, k, l it is true that $\overline{A}_i(t,t) \doteq \overline{A}_j(t,t) \doteq \overline{B}_k(t,t) \doteq \overline{B}_l(t,t)$, where $f(t) \doteq g(t)$

means that $f(t) = c \cdot g(t)$, where $c \neq 0$ is a constant. We leave the proof of this to the reader. This simple fact allows us to prove a large number of 'weak' invariants for rational functions of any degree.

The following simple unproven algorithm which uses only resultants will compute far more weak invariants than the algorithm in Section 5. This algorithm is a natural extension of Section 5. We have not seen this algorithm fail. As an illustration of the algorithm, define $\rho_s(A_i(s,t), A_j(s,t)) = F(t)$, where $i \neq j$ and ρ_s is the resultant with respect to s. In other words, $\rho_s(A_i(s,t)A_j(s,t))$ eliminates the variable s to give a polynomial in t. Also, $\rho_s(A_k(s,t), B_l(s,t)) = G(t)$. It is reasonably easy to show that $B_n(s,t) = B_n(t)$. That is, $B_n(s,t)$ is a polynomial in t only. Also, it can be shown that $B_n(t)$ divides both $F(t)$ and $G(t)$. Suppose $F(t) = B_n(t) \cdot \overline{F}(t)$ and $G(t) = B_n(t) \cdot \overline{G}(t)$, where $\overline{F}, \overline{G}$ are polynomials whose coefficients are polynomials in $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n$. Then $\rho(B_n(t), \overline{G}(t)), \rho(\overline{F}(t), B_n(t))$ and $\rho(\overline{F}(t), \overline{G}(t))$ are all weak invariants. Furthermore, suppose $F(t) = \overline{F}_1(t) \cdot \overline{F}_2(t) \cdot \cdots \cdot \overline{F}_k(t)$, where $\overline{F}_1(t) = B_n(t)$ and $G(t) = \overline{G}_1(t) \cdot \overline{G}_2(t) \cdot \cdots \cdot \overline{G}_l(t)$, where $\overline{G}_1(t) = B_n(t)$, is the complete or partial factorization of $F(t), G(t)$ into polynomials whose coefficients are polynomials in $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n$. Then each $\rho(\overline{F}_i(t), \overline{G}_j(t))$ is a weak invariant. Of course, $\rho(\overline{F_1}(t), \overline{G_1}(t)) = \rho(B_n(t), B_n(t)) = 0$. Also, each $\rho(\overline{F}_i(t), \overline{F}_j(t)), i \neq j$, is a weak invariant and each $\rho(\overline{G}_i H), \overline{G}_j H), i \neq j$ is a weak invariant. Suppose next that some $\overline{F}_i(t)$ or $\overline{G}_j(t)$ is not a primitive polynomial, and suppose $\theta(A_1, \ldots, A_n, B_1, \ldots, B_n)$ is a polynomial that divides all the polynomial coefficients of $\overline{F}_i(t)$ or $\overline{G}_j(t)$. Then $\theta(A_1, \ldots, A_n, B_1, \ldots, B_n)$ is also a weak invariant. Using the above factorizations $F(t) = \overline{F}_1(t) \cdot \overline{F}_2(t) \cdots \overline{F}_k(t)$, where $\overline{F}_1(t) = B_n(t)$ and $G(t) = \overline{G}_1(t) \cdot \overline{G}_2(t) \cdots \overline{G}_l(t)$, where $\overline{G}_1(t) = B_n(t)$, it is also true that each $D(\overline{F}_i(t))$ and each $D(\overline{G}_i(t))$ is an invariant. D denotes the discriminant. (See Definition 4.)

Note that it is easily proved from the theory of resultants that if $P_1(x), \ldots, P_n(x)$, $Q_1(x), \ldots, Q_m(x)$ are any polynomials, then

$$\rho(\prod_{i=1}^{n} P_i(x), \prod_{j=1}^{m} Q_j(x) = \prod \rho(P_i(x), Q_j(x))$$

and

$$D(\prod_{i=1}^{n}(P_i(x)) = \prod_{i=1}^{n} D(P_i(x)) \cdot \left[\prod_{i>j} \rho(P_i(x), P_j(x))\right]^2 .$$

See Lemma 11. These facts makes the unproven algorithm seem natural.

6 Finding Some First Level Invariants

We now apply the General Principle and some analogous reasoning to find three first level invariants of $\frac{P(x)}{Q(x)} \rightarrow \frac{\overline{P}(x)}{\overline{Q}(x)}$.

To see the general pattern, we consider the simple case where $\left(\frac{ax+b}{cx+d}\right)^{-1} \circ \frac{Ax^2+Bx+C}{Hx^2+Dx+E} \circ \left(\frac{ax+b}{cx+d}\right) = \frac{\overline{A}x^2+\overline{B}x+\overline{C}}{\overline{H}x^2+\overline{D}x+\overline{E}}$.

If we study the action in this transformation, we can easily derive three weak invariants that are simpler than what you might expect in the general method of Section 5 in which we used three coefficients $\overline{A}_i(s,t), \overline{A}_j(s,t), \overline{B}_j(s,t)$.

First, we observe that if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, then Ax^2+Bx+C and Hx^2+Dx+E are relatively prime if and only if $\overline{A}x^2+\overline{B}x+\overline{C}, \overline{H}x^2+Dx+\overline{E}$ are relatively prime. This means that

$$\rho\left(Ax^2+Bx+C, Hx^2+Dx+E\right) = 0$$

if and only if $\rho\left(\overline{A}x^2+\overline{B}x+\overline{C}, \overline{H}x^2+\overline{D}x+\overline{E}\right) = 0$, where ρ is the resultant. (See Definition 4.)

Suppose Ax^2+Bx+C and Hx^2+Dx+E are fixed and $\rho(Ax^2+Bx+C, Hx^2+Dx+E) \neq 0$. Then, if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, it follows that $\rho(\overline{A}x^2+\overline{B}x+\overline{C}, \overline{H}x^2+\overline{D}x+\overline{E}) \neq 0$, where $\rho(\overline{A}x^2+\overline{B}x+\overline{C}, \overline{H}x^2+\overline{D}x+\overline{E})$ is a polynomial in a, b, c, d. From Lemma 3, we know that $\rho(\overline{A}x^2+\overline{B}x+\overline{C}, \overline{H}x^2+\overline{D}x+\overline{E}) = \overline{c} \cdot \begin{vmatrix} a & b \\ c & d \end{vmatrix}^n, \overline{c} \neq 0$. Using $a = 1, b = c = 0, d = 1$ we see that $(*)$ $\qquad \rho(\overline{A}x^2+\overline{B}x+\overline{C}, \overline{H}x^2+\overline{D}x+\overline{E}) = \rho(Ax^2+Bx+C, Hx^2+Dx+E) \cdot \begin{vmatrix} a & b \\ c & d \end{vmatrix}^6$, where $n = 6$ is evaluated as in Section 5. Of course, $(*)$ is automatically true when $\rho(Ax^2+Bx+C, Hx^2+Dx+E) = 0$. Lemma 3 proves $(*)$ for us.

However, in Section 9 we give another proof of this for arbitrary degree rational functions by using the basic properties of resultants. This invariant can also be derived by considering $\overline{A}(s,t) = 0, \overline{H}(s,t) = 0$. Therefore the invariant is a true first level invariant.

Also, in this example we compute

$$\begin{aligned} \overline{H} &= Ha^3 + (D-A)\,a^2c + (E-B)\,ac^2 - Cc^3 \\ &= c^3\left[Ht^3 + (D-A)\,t^2 + (E-B)\,t - C\right]. \end{aligned}$$

Also,

$$\begin{aligned} \overline{D} &= d\left[Da^2 + (2E-B)\,ac - 2Cc^2\right] + b\left[2Ha^2 + (-2A+D)\,ac - Bc^2\right] \\ &= c^2d\left[\left(Dt^2 + (2E-B)\,t - 2C\right) + s\left(2Ht^2 + (-2A+D)\,t - B\right)\right] \end{aligned}$$

where as always we are calling $\frac{a}{c} = t, \frac{b}{d} = s$.

We now define three functions.

$$\begin{aligned} H(t) &= Ht^3 + (D-A)t^2 + (E-B)t - C, \\ F(t) &= Dt^2 + (2E-B)t - 2C, \\ G(t) &= 2Ht^2 + (-2A+D)t - B. \end{aligned}$$

Of course, $\overline{H} = c^3 H(t)$ and $\overline{D} = c^2 d[F(t) + sG(t)]$.

Since $P(x) = Ax^2 + Bx + C, Q(x) = Hx^2 + Dx + E$, we see that $H(x) = xQ(x) - P(x)$ and $G(x) = xQ'(x) - P'(x)$. These definitions make sense when $P(x), Q(x)$ have an arbitrary degree n. In general, $\overline{A}_0(s,t) = \overline{A}_0(s)$ and $\overline{B}_n(s,t) = \overline{B}_n(t)$ are the only two polynomials $\overline{A}_i(s,t), \overline{B}_i(s,t)$ that are polynomials of a single variable, and we are going to take care of these once and for all. Also, $\overline{B}_n(t)$ and $\overline{A}_0(s)$ lead to the same invariant. Although it is not exactly proved by the General Principle, unless we augment it slightly, we strongly suspect that whether $\overline{B}_n(t) = H(t)$ has or does not have repeated roots is an invariant under $\frac{P(x)}{Q(x)} \rightarrow \frac{\overline{P}(x)}{\overline{Q}(x)}$. This suspicion is correct, and we will soon prove this for rational functions $\frac{P(x)}{Q(x)} \rightarrow \frac{\overline{P}(x)}{\overline{Q}(x)}$ of arbitrary degree by using $D(H(t))$, the discriminant of $H(t)$.

We next observe that $F(t) + tG(t) = 2H(t)$. Again, observe that $\overline{H} = c^3 H(t)$ and $\overline{D} = c^2 d[F(t) + sG(t)]$. We now show that if $t \neq s$ then $H(t) = 0$ and $F(t) + sG(t) = 0$ if and only if $H(t) = 0$ and $G(t) = 0$. Now if $G(t) = H(t) = 0$, then $F(t) = 0$ which implies that $F(t) + sG(t) = 0$.

Next, suppose $H(t) = 0, F(t) + sG(t) = 0, s \neq t$. We now show that $G(t) = 0$.

Now, $F(t) = 2H(t) - tG(t)$. Therefore, $F(t) + sG(t) = 0$ implies $[2H(t) - tG(t)] + sG(t) = 2H(t) + (s-t)G(t) = 0$. Since $H(t) = 0$, $s \neq t$, this implies $G(t) = 0$. Therefore, we see that $H(t) = 0, F(t) + sG(t) = 0, s \neq t$ is true if and only if $H(t) = G(t) = 0, s \neq t$.

Of course, $H(t) = 0, G(t) = 0$ have a common solution t if and only if $\frac{1}{H}\rho(G(t), H(t)) = 0$. The General Principle of Section 5 tells us that this resultant $\frac{1}{H}\rho(G(t), H(t))$ leads to an invariant.

Note 3 If we deal with the transformation $(ax+b)^{-1} \circ \frac{P(x)}{Q(x)} \circ (ax+b) = \frac{\overline{P}(x)}{\overline{Q}(x)}, a \neq 0$, then the mathematics simplifies considerably, and we can apply reasoning that is analogous to the General Principle of Section 5 to define a vast number of first level invariants that involve the two variables a, b.

7 Stating the Three Invariants

As always, $P(x) = \sum_{i=0}^{n} A_i x^i, A_n \neq 0, Q(x) = \sum_{i=0}^{n} B_i x^i, B_n \neq 0$. We also assume that $P(x), Q(x)$ have no roots in common.

As always, $\left(\frac{ax+b}{cx+d}\right)^{-1}\circ\frac{P(x)}{Q(x)}\circ\left(\frac{ax+b}{cx+d}\right)=\frac{\overline{P}(x)}{\overline{Q}(x)}$ where $\overline{P}(x)=dP\left(\left(\frac{ax+b}{cx+d}\right)\right)-bQ\left(\left(\frac{ax+b}{cx+d}\right)\right), \overline{Q}(x)=-cP\left(\left(\frac{ax+b}{cx+d}\right)\right)+aQ\left(\left(\frac{ax+b}{cx+d}\right)\right)$.

Also,

$$\begin{aligned}
H(x) &= xQ(x)-P(x),\\
G(x) &= xQ'(x)-P'(x),\\
\overline{H}(x) &= x\overline{Q}(x)-\overline{P}(x),\\
\overline{G}(x) &= x\overline{Q}'(x)-\overline{P}'(x).
\end{aligned}$$

Let $\overline{H}_{n+1}, H_{n+1}$ be the leading coefficients of $\overline{H}(x), H(x)$. Also, $\rho(P(x), Q(x))$ denotes the standard resultant of two polynomials and $D(P(x))$ is the standard discriminant of a single polynomial. We also assume that all polynomials have the degree that they should have.

Theorem 1 $\rho\left(\overline{P}(x), \overline{Q}(x)\right)=\begin{vmatrix} a & b \\ c & d \end{vmatrix}^{n(n+1)}\cdot\rho(P(x), Q(x))$.

Theorem 2 $D\left(\overline{H}(x)\right)=\begin{vmatrix} a & b \\ c & d \end{vmatrix}^{n(n+1)}\cdot D(H(x))$.

Theorem 3 $\frac{1}{\overline{H}_{n+1}}\rho\left(\overline{G}(x), \overline{H}(x)\right)=\begin{vmatrix} a & b \\ c & d \end{vmatrix}^{n(n+1)}\cdot\frac{1}{H_{n+1}}\rho(G(x), H(x))$.

Observation 2 Of course, $\rho(P(x), Q(x))\neq 0$ and $\rho\left(\overline{P}(x), \overline{Q}(x)\right)\neq 0$ since $P(x), Q(x)$ are relatively prime and $\overline{P}(x), \overline{Q}(x)$ are relatively prime. Therefore, if we divide the invariants of Theorems 2, 3 by the invariant of Theorem 1, then we have two 'strong' invariants that are independent of a, b, c, d. Also, since the numerator and denominator are homogeneous polynomials in the same total degrees, these two 'strong' invariants are also invariants when we define $\frac{P(x)}{Q(x)}=\frac{tP(x)}{tQ(x)}, t\neq 0$, and define $\frac{\overline{P}(x)}{\overline{Q}(x)}=\frac{\overline{k}\overline{P}(x)}{\overline{k}\overline{Q}(x)}, \overline{k}\neq 0$. The reader should compare this statement with Notes 1, 2.

We now get down to the nuts and bolts and prove these three theorems on their level by using some very basic properties of resultants and discriminants.

8 Basic Properties of Resultants and Discriminants

Let $P(x)=\sum_{i=0}^{n}A_ix^i=A_n\cdot\prod_{i=1}^{n}(x-r_i), A_n\neq 0, Q(x)=\sum_{i=0}^{m}B_ix^i=B_m\cdot\prod_{i=1}^{m}(x-s_i), B_m\neq 0.$

Sometimes we restrict $n=m$ and sometimes we do not.

Definition 4 *The resultant* $\rho(P(x), Q(x)) = A_n^m B_m^n \prod (r_i - s_j)$.

Also, the discriminant $D(P(x)) = (-1)^{\frac{1}{2}n(n-1)} \left(\frac{1}{A_n}\right) \rho(P(x), P'(x))$.

Lemma 4 $D(P(x)) = A_n^{2n-2} \prod_{i>j} (r_i - r_j)^2$.

Proof. The proof is standard ■

The following axiom will allow a naive person to read this paper.

Axiom 1 $\rho(P(x), Q(x))$ *equals the determinant of the* $(n+m) \times (n+m)$ *matrix* M *defined as follows.*

Each row $1 \leq i \leq m$ *of* M *is defined as follows.*

$\overbrace{0, 0, \cdots, 0}^{\leftarrow i-1 \rightarrow} A_n, A_{n-1}, \cdots, A_0, \overbrace{0, 0, \cdots, 0}^{\leftarrow m-i \rightarrow}$.

Each row $m+i, 1 \leq i \leq n$ *of* M *is defined as follows.*

$\overbrace{0, 0, \cdots, 0}^{\leftarrow i-1 \rightarrow}, B_m, B_{m-1}, \cdots, B_0, \overbrace{0, 0, \cdots, 0}^{\leftarrow n-i \rightarrow}$.

See pp. 99-104, [1]. The following lemmas can be proved from Definition 4, Lemma 4 and Axiom 1, and they are sufficient to prove Theorems 1, 2, 3.

Lemma 5 $P(x) = \sum_{i=0}^{n} A_i x^i = A_n \cdot \prod_{i=1}^{n} (x - r_i), \; A_n \neq 0.$

Then $\prod_{i=1}^{n} (cr_i - a) = \frac{(-c)^n}{A_n} \cdot P\left(\frac{a}{c}\right), c \neq 0.$

Lemma 6 *Using the notation of Definition 1, define*

$$\begin{aligned}
P(x) &= \sum_{i=0}^{n} A_i x^i = A_n \cdot \prod_{i=1}^{n} (x - r_i), A_n \neq 0, \\
\overline{P}(x) &= P\left(\left(\frac{ax+b}{cx+d}\right)\right) = \sum_{i=0}^{n} \overline{A}_i x^i, \overline{A}_n \neq 0, \\
Q(x) &= \sum_{i=0}^{m} B_i x^i = B_m \cdot \prod_{i=1}^{m} (x - s_i), \; B_m \neq 0, \\
\overline{Q}(x) &= Q\left(\left(\frac{ax+b}{cx+d}\right)\right) = \sum_{i=0}^{m} \overline{B}_i x^i, \; \overline{B}_m \neq 0,
\end{aligned}$$

where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$

Then $\rho\left(\overline{P}(x), \overline{Q}(x)\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}^{nm} \cdot \rho(P(x), Q(x))$.

Note 4 As always, in Lemmas 6, 7 we consider a, b, c, d to be fixed and $P\left(\left(\frac{ax+b}{cx+d}\right)\right), Q\left(\left(\frac{ax+b}{cx+d}\right)\right)$ must be computed exactly as in Definition 1. Also, as always, the cases $\overline{A}_n = 0$ or $\overline{B}_m = 0$ can be handled by continuity, but $P\left(\left(\frac{ax+b}{cx+d}\right)\right), Q\left(\left(\frac{ax+b}{cx+d}\right)\right)$ are still considered to be of degrees n, m.

Proof of Lemma 6 *Let* $\overline{P}(x) = \overline{A}_n \cdot \prod_{i=1}^{n} (x - \overline{r}_i), \overline{Q}(x) = \overline{B}_m \cdot \prod_{i=1}^{m} (x - \overline{s}_i)$. *We will assume* $c \neq 0$ *and handle* $c = 0$ *by continuity.*

Now

$$\begin{aligned}
\overline{P}(x) &= P\left(\left(\frac{ax+b}{cx+d}\right)\right) = \sum_{i=0}^{n} A_i (ax+b)^i (cx+d)^{n-i} \\
&= \overline{A}_n \cdot \prod_{i=1}^{n} (x - \overline{r}_i) = \left(\sum_{i=0}^{n} A_i a^i c^{n-i}\right) \cdot \prod_{i=1}^{n} (x - \overline{r}_i) \\
&= c^n P\left(\frac{a}{c}\right) \cdot \prod_{i=1}^{n} (x - \overline{r}_i).
\end{aligned}$$

Also,

$$\begin{aligned}
\overline{Q}(x) &= Q\left(\left(\frac{ax+b}{cx+d}\right)\right) = \sum_{i=0}^{m} B_i (ax+b)^i (cx+d)^{m-i} \\
&= \overline{B}_m \cdot \prod_{i=1}^{m} (x - \overline{s}_i) = \left(\sum_{i=0}^{m} B_i a^i c^{m-i}\right) \cdot \prod_{i=1}^{m} (x - \overline{s}_i) \\
&= c^m Q\left(\frac{a}{c}\right) \cdot \prod_{i=1}^{m} (x - \overline{s}_i).
\end{aligned}$$

Now $\rho(P, Q) = A_n^m A_m^n \Pi(r_i - s_j)$ *and* $\rho\left(\overline{P}, \overline{Q}\right) = \overline{A}_n^m \overline{B}_m^n \Pi(\overline{r}_i - \overline{s}_j)$ *where* $\overline{A}_n = c^n P\left(\frac{a}{c}\right), \overline{B}_m = c^m Q\left(\frac{a}{c}\right)$.

Also, $\overline{r}_i = \frac{b - dr_i}{cr_i - a}, \overline{s}_i = \frac{b - ds_i}{cs_i - a}$.

Therefore, $\overline{r}_i - \overline{s}_j = \frac{(ad-bc)(r_i - s_j)}{(cr_i - a)(cs_j - a)}$.

Now, $\rho\left(\overline{P}, \overline{Q}\right) = \overline{A}_n^m \overline{B}_m^n \Pi\left[\frac{(ad-bc)(r_i-s_j)}{(cr_i-a)(cs_j-a)}\right] = \frac{\overline{A}_n^m \overline{B}_m^n (ad-bc)^{nm} \Pi(r_i - s_j)}{\left[\frac{(-c)^n}{A_n} P\left(\frac{a}{c}\right)\right]^m \cdot \left[\frac{(-c)^m}{B_m} Q\left(\frac{a}{c}\right)\right]^n}$, *which, by Lemma 5* $= \frac{A_n^m \overline{A}_n^m B_m^n \overline{B}_m^n (ad-bc)^{nm} \Pi(r_i - s_j)}{\overline{A}_n^m \overline{B}_m^n} = (ad-bc)^{nm} A_n^m B_m^n \Pi(r_i - s_j) = (ad-bc)^{nm} \rho(P(x), Q(x))$.■

Lemma 7 $P(x) = \sum_{i=0}^{n} A_i x^i, A_n \neq 0,$ *and* $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$. *Then*

$$D\left(P\left(\left(\frac{ax+b}{cx+d}\right)\right)\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}^{n(n-1)} D(P(x)).$$

Proof. The proof which uses Lemma 4 is very similar to the proof of Lemma 6 and is left to the reader. ■

Lemma 8 *Suppose* $P(x), Q(x), \overline{P}(x), \overline{Q}(x)$ *are all* n*th degree polynomials and* $\overline{P}(x) = aP(x) + bQ(x), \overline{Q}(x) = cP(x) + dQ(x), \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$. *Then*

$$\rho\left(\overline{P}(x), \overline{Q}(x)\right) = \rho(aP(x) + bQ(x), cP(x) + dQ(x)) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}^n \cdot \rho(P(x), Q(x)).$$

Proof. We assume $d \neq 0$ and as always we handle $d = 0$ by continuity. The case $n = 2$ easily shows the pattern. Now

$$\begin{aligned} & \rho(aP(x) + bQ(x), cP(x) + dQ(x)) \\ = & \begin{vmatrix} aA_2 + bB_2 & aA_1 + bB_1 & aA_0 + bB_0 & 0 \\ 0 & aA_2 + bB_2 & aA_1 + bB_1 & aA_0 + bB_0 \\ cA_2 + dB_2 & cA_1 + dB_1 & cA_0 + dB_0 & 0 \\ 0 & cA_2 + dB_2 & cA_1 + dB_1 & cA_0 + dB_0 \end{vmatrix} \\ = & \begin{vmatrix} \left(\frac{ad-bc}{d}\right) A_2 & \left(\frac{ad-bc}{d}\right) A_1 & \left(\frac{ad-bc}{d}\right) A_0 & 0 \\ 0 & \left(\frac{ad-bc}{d}\right) A_2 & \left(\frac{ad-bc}{d}\right) A_1 & \left(\frac{ad-bc}{d}\right) A_0 \\ cA_2 + dB_2 & cA_1 + dB_1 & cA_0 + dB_0 & 0 \\ 0 & cA_2 + dB_2 & cA_1 + dB_1 & cA_0 + dB_0 \end{vmatrix} \\ = & \begin{vmatrix} \left(\frac{ad-bc}{d}\right) A_2 & \left(\frac{ad-bc}{d}\right) A_1 & \left(\frac{ad-bc}{d}\right) A_0 & 0 \\ 0 & \left(\frac{ad-bc}{d}\right) A_2 & \left(\frac{ad-bc}{d}\right) A_1 & \left(\frac{ad-bc}{d}\right) A_0 \\ dB_2 & dB_1 & dB_0 & 0 \\ 0 & dB_2 & dB_1 & dB_0 \end{vmatrix} \\ = & \begin{vmatrix} a & b \\ c & d \end{vmatrix}^2 \cdot \rho(P(x), Q(x)). \qquad ■ \end{aligned}$$

Lemma 9 *Suppose* $P(x)$ *is an* n*th degree polynomial and* $Q(x)$ *is an* m*th degree polynomial. Also,* $c \neq 0, \overline{c} \neq 0$. *Then* $\rho(cP(x), \overline{c}Q(x)) = c^m \overline{c}^n \rho(P(x), Q(x))$

Lemma 10 $P(x) = \sum_{i=0}^{n+1} A_i x^i, A_{n+1} \neq 0,$ *and* $Q(x) = \sum_{i=0}^{n} B_i x^i, B_n \neq 0$. *Also,* $c \neq 0$.
Then $\rho(Q + cP, P) = (-1)^{n+1} A_{n+1} \rho(Q, P)$.

Proof. We note that $Q + cP$ and P are of degree $n + 1$ and Q is of degree n. Thus, $\rho(Q + cP, P)$ is evaluated by a $(2n + 2) \times (2n + 2)$ determinant, and $\rho(Q, P)$ is evaluated by a $(2n + 1) \times (2n + 1)$ determinant. The proof is very similar to Lemma 8 and is a trivial application of Axiom 1. Also, the special case $n = 2$ easily shows the pattern and the reader can supply the details. ■

Lemma 11 $P(x) = \sum_{i=0}^{n} A_i x^i = A_n \cdot \prod_{i=1}^{n}(x - r_i), A_n \neq 0, Q(x) = \sum_{i=0}^{m} B_i x^i = B_m \cdot \prod_{i=1}^{m}(x - s_i), B_m \neq 0.$

Also, $a \neq 0$. Then $\rho((ax+b)P(x), Q(x)) = a^m Q\left(\frac{-b}{a}\right)\rho(P(x), Q(x))$.

Proof. $\rho((ax+b)P(x), Q(x)) = (aA_n)^m B_m^{n+1}\left[\prod_{i=1}^{m}\left(\frac{-b}{a} - s_i\right)\right]\left[\prod(r_i - s_j)\right]$.

Now $Q\left(\frac{-b}{a}\right) = B_m \prod_{i=1}^{m}\left(\frac{-b}{a} - s_i\right)$. Therefore, $\prod_{i=1}^{m}\left(\frac{-b}{a} - s_i\right) = \frac{Q\left(\frac{-b}{a}\right)}{B_m}$. Therefore,

$$\begin{aligned}
\rho((ax+b)P(x), Q(x)) &= \frac{(aA_n)^m B_m^{n+1} Q\left(\frac{-b}{a}\right)\Pi(r_i - s_j)}{B_m} \\
&= a^m Q\left(\frac{-b}{a}\right) A_n^m B_m^n \Pi(r_i - s_j) \\
&= a^m Q\left(\frac{-b}{a}\right)\rho(P(x), Q(x)). \qquad \blacksquare
\end{aligned}$$

Corollary 1 *Suppose $R(x)$ is a polynomial of degree $n+1$ and $(ax+b) \mid R(x)$ where $a \neq 0$. Also, $Q(x)$ is a polynomial of degree m. Then $\rho\left(\frac{R(x)}{ax+b}, Q(x)\right) = \frac{\rho(R(x),Q(x))}{a^m Q\left(\frac{-b}{a}\right)}$.*

Proof.

$$\begin{aligned}
\rho(R(x), Q(x)) &= \rho\left((ax+b)\left(\frac{R(x)}{ax+b}\right), Q(x)\right) \\
&= a^m Q\left(\frac{-b}{a}\right)\rho\left(\frac{R(x)}{ax+b}, Q(x)\right). \qquad \blacksquare
\end{aligned}$$

Lemma 11 can be generalized to $\rho(\prod_{i=1}^{n} P_i(x), \prod_{i=1}^{m} Q_i(x)) = \prod \rho(P_i(x), Q_j(x))$. But this is not needed.

9 Proving Theorems 1, 2, and 3

The definitions $P(x), Q(x), \overline{P}(x), \overline{Q}(x), H(x), \overline{H}(x), G(x), \overline{G}(x)$ are given in Section 7.

Theorem 1. $\rho\left(\overline{P}(x), \overline{Q}(x)\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}^{n(n+1)} \cdot \rho(P(x), Q(x))$.

Proof. Using the definitions of $\overline{P}(x), \overline{Q}(x)$,

$$\rho\left(\overline{P}(x), \overline{Q}(x)\right) = \rho\left(dP\left(\left(\frac{ax+b}{cx+d}\right)\right) - bQ\left(\left(\frac{ax+b}{cx+d}\right)\right), -cP\left(\left(\frac{ax+b}{cx+d}\right)\right) + aQ\left(\left(\frac{ax+b}{cx+d}\right)\right)\right),$$

which, by Lemma 8, equals $\begin{vmatrix} d & -b \\ -c & a \end{vmatrix}^n \cdot \rho\left(P\left(\left(\frac{ax+b}{cx+d}\right)\right), Q\left(\left(\frac{ax+b}{cx+d}\right)\right)\right)$, which, by Lemma 6, equals $\begin{vmatrix} a & b \\ c & d \end{vmatrix}^n \cdot \begin{vmatrix} a & b \\ c & d \end{vmatrix}^{n^2} \cdot \rho\left(P\left(x\right), Q\left(x\right)\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}^{n(n+1)} \cdot \rho\left(P\left(x\right), Q\left(x\right)\right)$. ■

Theorem 2. $D\left(\overline{H}\left(x\right)\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}^{n(n+1)} \cdot D\left(H\left(x\right)\right)$.

Proof. Now $H\left(x\right) = xQ\left(x\right) - P\left(x\right)$ and $\overline{H}\left(x\right) = x\overline{Q}\left(x\right) - \overline{P}\left(x\right)$. From the definitions of $\overline{P}\left(x\right), \overline{Q}\left(x\right)$ in Lemma 1, we know $(*)$ $\overline{H}\left(x\right) = -\left(cx+d\right) P\left(\left(\frac{ax+b}{cx+d}\right)\right) + \left(ax+b\right) Q\left(\left(\frac{ax+b}{cx+d}\right)\right)$.

Now $H\left(x\right)$ is of degree $n+1$. Also, $\overline{H}\left(x\right)$ is of degree $n+1$ unless it is degenerate which as always we can handle by continuity.

We now show that $\overline{H}\left(x\right) = H\left(x\right) \circ \left(\left(\frac{ax+b}{cx+d}\right)\right) = H\left(\left(\frac{ax+b}{cx+d}\right)\right)$. Then from Lemma 7 we will know that $D\left(\overline{H}\left(x\right)\right) = DH\left(\left(\frac{ax+b}{cx+d}\right)\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}^{n(n+1)} \cdot D\left(H\left(x\right)\right)$ which will complete the proof. Now

$$\begin{aligned}
H\left(\left(\frac{ax+b}{cx+d}\right)\right) &= H\left(x\right) \circ \left(\left(\frac{ax+b}{cx+d}\right)\right) \\
&= \left(cx+d\right)^{n+1} \cdot H\left(\frac{ax+b}{cx+d}\right) \\
&= \left(cx+d\right)^{n+1} \cdot \left[\frac{ax+b}{cx+d} Q\left(\frac{ax+b}{cx+d}\right) - P\left(\frac{ax+b}{cx+d}\right)\right] \\
&= -\left(cx+d\right)^{n+1} P\left(\frac{ax+b}{cx+d}\right) + \left(ax+b\right)\left(cx+d\right)^n Q\left(\frac{ax+b}{cx+d}\right) \\
&= -\left(cx+d\right) P\left(\left(\frac{ax+b}{cx+d}\right)\right) + \left(ax+b\right) Q\left(\left(\frac{ax+b}{cx+d}\right)\right) \\
&= \overline{H}\left(x\right) \text{ from } (*). \qquad ■
\end{aligned}$$

Theorem 3 is much harder to prove than Theorems 1 and 2 which were almost trivial to prove. But we point out that we are dealing with rational functions of arbitrary degree n. So it is worth the extra work.

Theorem 3.

$$\begin{aligned}
&\frac{1}{\overline{H}_{n+1}} \rho\left(\overline{G}\left(x\right), \overline{H}\left(x\right)\right) \\
&= \begin{vmatrix} a & b \\ c & d \end{vmatrix}^{n(n+1)} \cdot \frac{1}{H_{n+1}} \rho\left(G\left(x\right), H\left(x\right)\right).
\end{aligned}$$

Note that $G\left(x\right), \overline{G}\left(x\right)$ are of degree n, and $H\left(x\right), \overline{H}\left(x\right)$ are of degree $n+1$. As always, degenerate cases are handled by continuity.

Proof. As always.

$$\begin{aligned}
\overline{P}(x) &= dP\left(\left(\frac{ax+b}{cx+d}\right)\right)-bQ\left(\left(\frac{ax+b}{cx+d}\right)\right), \\
\overline{Q}(x) &= -cP\left(\left(\frac{ax+b}{cx+d}\right)\right)+aQ\left(\left(\frac{ax+b}{cx+d}\right)\right)
\end{aligned}$$

Also, $H(x)=xQ(x)-P(x)\,, G(x)=xQ'(x)-P'(x)\,.$

Also, $\overline{H}(x)=x\overline{Q}(x)-\overline{P}(x)=-(cx+d)\,P\left(\left(\frac{ax+b}{cx+d}\right)\right)+(ax+b)\,Q\left(\left(\frac{ax+b}{cx+d}\right)\right).$

Now $\overline{P}(x)=d(cx+d)^n P\left(\frac{ax+b}{cx+d}\right)-b(cx+d)^n Q\left(\frac{ax+b}{cx+d}\right), \overline{Q}(x)=-c(cx+d)^n P\left(\frac{ax+b}{cx+d}\right)+a(cx+d)^n Q\left(\frac{ax+b}{cx+d}\right)$. We now calculate $\overline{G}(x)=x\overline{Q}'(x)-\overline{P}'(x)$. Now $\left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$.

By straightforward calculation,

$$\overline{P}'(x)=\frac{1}{cx+d}\left[\begin{array}{c} ncdP\left(\left(\frac{ax+b}{cx+d}\right)\right) \\ +d(ad-bc)\,P'\left(\left(\frac{ax+b}{cx+d}\right)\right) \\ -nbcQ\left(\left(\frac{ax+b}{cx+d}\right)\right) \\ -b(ad-bc)\,Q'\left(\left(\frac{ax+b}{cx+d}\right)\right)\end{array}\right].$$

Note that $P\left(\left(\frac{ax+b}{cx+d}\right)\right), Q\left(\left(\frac{ax+b}{cx+d}\right)\right)$ are of degree n and $P'\left(\left(\frac{ax+b}{cx+d}\right)\right), Q'\left(\left(\frac{ax+b}{cx+d}\right)\right)$ are of degree $n-1$.

Also, by calculation,

$$\overline{Q}'(x)=\frac{1}{cx+d}\left[\begin{array}{c} -nc^2P\left(\left(\frac{ax+b}{cx+d}\right)\right) \\ -c(ad-bc)\,P'\left(\left(\frac{ax+b}{cx+d}\right)\right) \\ +nacQ\left(\left(\frac{ax+b}{cx+d}\right)\right) \\ +a(ad-bc)\,Q'\left(\left(\frac{ax+b}{cx+d}\right)\right)\end{array}\right].$$

Therefore,

$$\begin{aligned}
\overline{G}(x) &= x\overline{Q}'(x)-\overline{P}'(x) \\
&= \frac{1}{cx+d}\left[\begin{array}{c} -nc(cx+d)\,P\left(\left(\frac{ax+b}{cx+d}\right)\right) \\ -(ad-bc)(cx+d)\,P'\left(\left(\frac{ax+b}{cx+d}\right)\right) \\ +nc(ax+b)\,Q\left(\left(\frac{ax+b}{cx+d}\right)\right) \\ +(ad-bc)(ax+b)\,Q'\left(\left(\frac{ax+b}{cx+d}\right)\right)\end{array}\right] \\
&= \frac{1}{cx+d}\left[nc\overline{H}(x)+(ad-bc)\,\overline{\theta}(x)\right] \text{ where} \\
\overline{\theta}(x) &= -(cx+d)\,P'\left(\left(\frac{ax+b}{cx+d}\right)\right)+(ax+d)\,Q'\left(\left(\frac{ax+b}{cx+d}\right)\right).
\end{aligned}$$

Of course, $\overline{\theta}(x)$ is a polynomial of degree n.

As always, $\overline{H}_{n+1}, H_{n+1}$ are the leading coefficients of $\overline{H}(x), H(x)$, and we need to show that

$$\frac{1}{\overline{H}_{n+1}}\rho\left(\overline{G}(x),\overline{H}(x)\right)=\left|\begin{array}{cc} a & b \\ c & d\end{array}\right|^{n(n+1)}\cdot\frac{1}{H_{n+1}}\rho\left(G(x),H(x)\right).$$

Now

$$\frac{1}{\overline{H}_{n+1}}\rho\left(\overline{G}\left(x\right),\overline{H}\left(x\right)\right)=\frac{1}{\overline{H}_{n+1}}\rho\left(\frac{1}{cx+d}\left[nc\overline{H}\left(x\right)+\left(ad-bc\right)\overline{\theta}\left(x\right)\right],\overline{H}\left(x\right)\right),$$

which by Corollary 1, $=\frac{1}{\overline{H}_{n+1}c^{n+1}\overline{H}\left(\frac{-d}{c}\right)}\left[\rho\left(nc\overline{H}\left(x\right)+\left(ad-bc\right)\overline{\theta}\left(x\right),\overline{H}\left(x\right)\right)\right]$, which by Lemma 10, $=\frac{(-1)^{n+1}\overline{H}_{n+1}}{\overline{H}_{n+1}c^{n+1}\overline{H}\left(\frac{-d}{c}\right)}\rho\left(\left(ad-bc\right)\overline{\theta}\left(x\right),\overline{H}\left(x\right)\right)$
$=\frac{(-1)^{n+1}(ad-bc)^{n+1}}{c^{n+1}\overline{H}\left(\frac{-d}{c}\right)}\rho\left(\overline{\theta}\left(x\right),\overline{H}\left(x\right)\right)=(**)$ where $(**)$ is computed as follows.

Now $\overline{H}\left(x\right)=-\left(cx+d\right)P\left(\left(\frac{ax+b}{cx+d}\right)\right)+\left(ax+b\right)Q\left(\left(\frac{ax+b}{cx+d}\right)\right).$

Also, $Q\left(\left(\frac{ax+b}{cx+d}\right)\right)=\sum_{i=0}^{n}B_i\left(ax+b\right)^i\left(cx+d\right)^{n-i}.$

Therefore, $\overline{H}\left(\frac{-d}{c}\right)=\frac{(-1)^{n+1}(ad-bc)^{n+1}B_n}{c^{n+1}}$.

Therefore, $(**)=\frac{1}{B_n}\rho\left(\overline{\theta}\left(x\right),\overline{H}\left(x\right)\right)=(***)$ where $\overline{\theta}\left(x\right)=-\left(cx+d\right)P'\left(\left(\frac{ax+b}{cx+d}\right)\right)+\left(ax+b\right)Q'\left(\left(\frac{ax+b}{cx+d}\right)\right)$ and $\overline{H}\left(x\right)=-\left(cx+d\right)P\left(\left(\frac{ax+b}{cx+d}\right)\right)+\left(ax+b\right)Q\left(\left(\frac{ax+b}{cx+d}\right)\right).$

We now show that $(***)=\frac{1}{B_n}\rho\left(G\left(\left(\frac{ax+b}{cx+d}\right)\right),H\left(\left(\frac{ax+b}{cx+d}\right)\right)\right).$

Then it will follow from Lemma 6 that $(***)=\begin{vmatrix} a & b \\ c & d \end{vmatrix}^{n(n+1)}\cdot\frac{1}{B_n}\rho\left(G\left(x\right),H\left(x\right)\right)=\begin{vmatrix} a & b \\ c & d \end{vmatrix}^{n(n+1)}\cdot\frac{1}{H_{n+1}}\rho\left(G\left(x\right),H\left(x\right)\right)$ which will complete the proof. Of course, $B_n=H_{n+1}$ since $H\left(x\right)=xQ\left(x\right)-P\left(x\right)$. So all we have to do is prove that $G\left(\left(\frac{ax+b}{cx+d}\right)\right)=\overline{\theta}\left(x\right),H\left(\left(\frac{ax+b}{cx+d}\right)\right)=\overline{H}\left(x\right)$. Now $H\left(\left(\frac{ax+b}{cx+d}\right)\right)=\overline{H}(x)$ was demonstrated in the proof of Theorem 2.

Now

$$\begin{aligned}G\left(\left(\frac{ax+b}{cx+d}\right)\right) &= \left(xQ'\left(x\right)-P'\left(x\right)\right)\circ\left(\left(\frac{ax+b}{cx+d}\right)\right)\\ &= \left(cx+d\right)^n\left[\frac{ax+b}{cx+d}Q'\left(\frac{ax+b}{cx+d}\right)-P'\left(\frac{ax+b}{cx+d}\right)\right]\\ &= \left(ax+b\right)Q'\left(\left(\frac{ax+b}{cx+d}\right)\right)-(cx+d)P'\left(\left(\frac{ax+b}{cx+d}\right)\right)\\ &= \overline{\theta}\left(x\right).\end{aligned}$$

Therefore $(***)=\frac{1}{B_n}\rho\left(G\left(\left(\frac{ax+b}{cx+d}\right)\right),H\left(\left(\frac{ax+b}{cx+d}\right)\right)\right)$. ■

10 Applications

Suppose $T\left(x\right)=\frac{Ax^2+Bx+C}{Hx^2+Dx+E}=\frac{P(x)}{Q(x)}$, $\overline{T}\left(x\right)=\frac{\overline{A}x^2+\overline{B}x+\overline{C}}{\overline{H}x^2+\overline{D}x+\overline{E}}=\frac{\overline{P}(x)}{\overline{Q}(x)}$ are rational quadratics. As in Observation 2, we combine Theorems 1, 2, and 3, to define two "strong" invariants. As always the invariants are computed by using Axiom 1 and assuming $A\neq 0,H\neq 0,\overline{A}\neq 0,\overline{H}\neq 0$. As always, continuity takes care of degeneracies.

Invariant $\theta = \frac{D(H(x))}{\rho(P(x),Q(x))}$ and Invariant $\phi = \frac{\left(\frac{1}{H}\right)\rho(G(x),H(x))}{\rho(P(x),Q(x))}$ where as always, $H(x) = xQ(x) - P(x)$ and $G(x) = xQ'(x) - P'(x)$. Of course, $\rho(P(x), Q(x)) \neq 0$ since $P(x), Q(x)$ have no roots in common. We also say that $T(x) = \frac{P(x)}{Q(x)}$ is a normal rational quadratic if $\theta\left(\frac{P(x)}{Q(x)}\right) \neq 0$. Note that θ, ϕ are also invariants under $\frac{P(x)}{Q(x)} = \frac{tP(x)}{tQ(x)}, t \neq 0, \frac{\overline{P}(x)}{\overline{Q}(x)} = \frac{k\overline{P}(x)}{k\overline{Q}(x)}, k \neq 0$, since θ, ϕ are each the ratio of two homogeneous polynomials of the same total degree 4. Suppose $T(x), \overline{T}(x)$ are normal rational quadratics. Then by using reasoning that is much simpler than the reasoning in this paper, we can show that $T(x) \sim \overline{T}(x)$ if and only if $(\theta, \phi) = \left(\overline{\theta}, \overline{\phi}\right)$.

Also, $T(x)$ is $\sim$ to a polynomial if and only if $\phi(T) = 0$.

After computing the invariants θ, ϕ of $\frac{Ax^2+Bx+C}{Hx^2+Dx+E}$ in terms of A, B, C, H, D, E, the reader might like to prove the following.

(1) $\frac{(a+c)x^2+2mx+cm}{x^2+2cx+(m-ac)} \sim \frac{x^2+s}{2x+t} \sim x^2$.

(2) $\frac{ax^2-2bx+c}{x^2-2ax+b} \sim \frac{x^2-2rx+t}{-2x+r} \sim \frac{1}{x^2}$.

In (1), (2), we assume that all rational quadratics are (genuine) non-reducible rational quadratics.

11 Discussion

This paper deals exclusively with first level invariants. By using the General Principle together with the fact that $\overline{A}_i(t,t) \doteq \overline{A}_j(t,t) \doteq \overline{B}_k(t,t) \doteq \overline{B}_l(t,t)$ for all i, j, k, l, we have computed and independently verified nine second level "weak" invariants for the rational quadratics $\frac{Ax^2+Bx+C}{Hx^2+Dx+E} \to \frac{\overline{A}x^2+\overline{B}x+\overline{C}}{\overline{H}x^2+\overline{D}x+\overline{E}}$. Combining these with the invariants computed by the unproven algorithm, we have roughly 30 interrelated weak invariants for rational quadratics and these have been independently verified. We estimate that the unproven algorithm will compute around 60 to 150 weak invariants for the 3rd degree rational functions. We believe there are a vast number of second level weak invariants for higher degree rational functions.

The reasoning in this paper also applies to each of the following. For transformation (d), we can prove an analogy of Theorem 1.

a. $P(x) \to \overline{P}(x) = P((\frac{ax+b}{cx+d}))$.

b. $\frac{P(x)}{Q(x)} \to \frac{\overline{P}(x)}{\overline{Q}(x)} = \frac{P(x)}{Q(x)} \circ (\frac{ax+b}{cx+d}) = \frac{P((\frac{ax+b}{cx+d}))}{Q((\frac{ax+b}{cx+d}))}$.

c. $\frac{P(x)}{Q(x)} \to \frac{\overline{P}(x)}{\overline{Q}(x)} = (\frac{\bar{a}x+\bar{b}}{\bar{c}x+\bar{d}}) \circ \frac{P(x)}{Q(x)}$.

d. $\frac{P(x)}{Q(x)} \to \frac{\overline{P}(x)}{\overline{Q}(x)} = (\frac{\bar{a}x+\bar{b}}{\bar{c}x+\bar{d}}) \circ \frac{P(x)}{Q(x)} \circ (\frac{ax+b}{cx+d})$.

e. $\frac{P(x)}{Q(x)} \to \frac{\overline{P}(x)}{\overline{Q}(x)} = (\frac{ax+b}{cx+d})^r \circ \frac{P(x)}{Q(x)} \circ (\frac{ax+b}{cx+d})^t$, where r, t are integers not both zero.

f. $P(x) \to \overline{P}(x) = P(x) \circ (ax+b) = P(ax+b)$.

g. $\frac{P(x)}{Q(x)} \to \frac{\overline{P}(x)}{\overline{Q}(x)} = (ax+b)^{-1} \circ \left(\frac{P(x)}{Q(x)}\right) \circ (ax+b)$,

and the list goes on.

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