Using Complex Weighted Centroids to Create Homothetic Polygons

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1 Abstract

After first defining weighted centroids that use complex arithmetic, we then make a simple observation which proves Theorem 1. We next define complex homothecy.

We then show how to apply this theory to triangles (or polygons) to create endless numbers of homothetic triangles (or polygon).

The first part of the paper is fairly standard. However, in the final part of the paper, we give two examples which illustrate that examples can easily be given in which the simple basic underpinning is so disguised that it is not at all obvious. Also, the entire paper is greatly enhanced by the use of complex arithmetic.

2 Introduction to the Basic Theory

Suppose A, B, C, x, y are complex numbers that satisfy $xA + yB = C, x + y = 1$. It easily follows that $A + y(B - A) = C$ and $x(A - B) + B = C$. This simple observation with its geometric interpretation is the basis of this paper.

Definition 1 *Suppose $M_1, M_2, \cdots, M_m$ are points in the complex plane and $k_1, k_2, \cdots, k_m$ are complex numbers that satisfy $\sum_{i=1}^{m} k_i = 1$. Of course, each complex point M_i is also a* *complex number. The weighted centroid of these complex points* $\{M_1, M_2, \cdots, M_m\}$ *with respect to* $\{k_1, k_2, \cdots, k_m\}$ *is a complex point* G_M *defined by* $G_M = \sum_{i=1}^{m} k_i M_i$.

The complex numbers $k_1, k_2, \cdots, k_m$ *are called weights and in the notation* G_M *it is always assumed that the reader knows what these weights are.*

If $k_1, k_2, \cdots, k_m, \overline{k}_1, \overline{k}_2, \cdots, \overline{k}_n$ *are complex numbers, we denote the sums* $S_k = \sum_{i=1}^{m} k_i$, $S_{\overline{k}} = \sum_{i=1}^{n} \overline{k}_i$.

Suppose $M_1, M_2, \cdots, M_m, \overline{M}_1, \overline{M}_2, \cdots, \overline{M}_n$ *are points in the complex plane. Also,* $k_1, k_2, \cdots, k_m, \overline{k}_1, \overline{k}_2, \cdots, \overline{k}_n$ *are complex numbers that satisfy* $\sum_{i=1}^{m} k_i + \sum_{i=1}^{n} \overline{k}_i = 1$. *Thus,* $S_k + S_{\overline{k}} = 1$.

Denote $G_{M \cup \overline{M}} = \sum_{i=1}^{m} k_i M_i + \sum_{i=1}^{n} \overline{k}_i \overline{M}_i$.

Thus, $G_{M \cup \overline{M}}$ is the weighted centroid of $\left\{M_1, \cdots, M_m, \overline{M}_1, \cdots, \overline{M}_n\right\}$ with respect to the weights $\left\{k_1, \cdots, k_m, \overline{k}_1, \cdots, \overline{k}_n\right\}$.

It is obvious that $\sum_{i=1}^{m} \frac{k_i}{S_k} = 1$ and $\sum_{i=1}^{n} \frac{\overline{k}_i}{S_{\overline{k}}} = 1$.

Denote $G_M = \sum_{i=1}^{m} \frac{k_i}{S_k} M_i$ and $G_{\overline{M}} = \sum_{i=1}^{n} \frac{\overline{k}_i}{S_{\overline{k}}} \overline{M}_i$.

Thus, G_M is the weighted centroid of $\{M_1, M_2, \cdots, M_m\}$ with respect to the weights $\left\{\frac{k_1}{S_k}, \frac{k_2}{S_k}, \cdots, \frac{k_m}{S_k}\right\}$ and $G_{\overline{M}}$ is the weighted centroid of $\left\{\overline{M}_1, \overline{M}_2, \cdots, \overline{M}_n\right\}$ with respect to the weights $\left\{\frac{\overline{k}_1}{S_{\overline{k}}}, \frac{\overline{k}_2}{S_{\overline{k}}}, \cdots, \frac{\overline{k}_n}{S_{\overline{k}}}\right\}$.

As always, these weights are understood in the notation $G_M, G_{\overline{M}}$.

Since $G_{M \cup \overline{M}} = \sum_{i=1}^{m} k_i M_i + \sum_{i=1}^{n} \overline{k}_i \overline{M}_i = S_k \cdot \sum_{i=1}^{m} \frac{k_i}{S_k} M_i + S_{\overline{k}} \cdot \sum_{i=1}^{n} \frac{\overline{k}_i}{S_{\overline{k}}} \overline{M}_i$ it is obvious that $(*)$ is true.

$(*)$ $S_k \cdot G_M + S_{\overline{k}} \cdot G_{\overline{M}} = G_{M \cup \overline{M}}$ where $S_k + S_{\overline{k}} = 1$.

From equation $(*)$ and $S_k + S_{\overline{k}} = 1$ it is easy to see that (1) and (2) are true.

(1) $G_M + S_{\overline{k}} \left(G_{\overline{M}} - G_M\right) \equiv G_{M \cup \overline{M}}$.

(2) $G_{\overline{M}} + S_k \left(G_M - G_{\overline{M}}\right) \equiv G_{M \cup \overline{M}}$.

3 Basic Theorem

The identity $(*)\, S_k \cdot G_M + S_{\overline{k}} \cdot G_{\overline{M}} = G_{M \cup \overline{M}}$, where $S_k + S_{\overline{k}} = 1$, and the formula (1) $G_M + S_{\overline{k}} \left(G_{\overline{M}} - G_M\right) = G_{M \cup \overline{M}}$ of Section 2 proves the following Theorem 1.

Theorem 1 Suppose $M_1, M_2, \cdots, M_m, \overline{M}_1, \overline{M}_2, \cdots, \overline{M}_n$ are points in the complex plane.

Also, suppose $P = \sum_{i=1}^{m} k_i M_i + \sum_{i=1}^{n} \overline{k}_i \overline{M}_i$ where $k_1, \cdots k_m, \overline{k}_1, \cdots, \overline{k}_n$ are complex numbers that satisfy $\sum_{i=1}^{m} k_i + \sum_{i=1}^{n} \overline{k}_i = 1$.

Then there exists complex numbers $x_1, x_2, \cdots, x_m$ where $\sum_{i=1}^{m} x_i = 1$ and there exists complex numbers $y_1, y_2, \cdots, y_n$ where $\sum_{i=1}^{n} y_i = 1$ and there exists a complex number z such that the following is true.

1. $x_1, \cdots, x_m, y_1, \cdots, y_n, z$ are rational function of $k_1, \cdots, k_m, \overline{k}_1, \cdots, \overline{k}_n$.
2. $P = Q + z(R - Q)$ where Q, R are defined by $Q = \sum_{i=1}^{m} x_i M_i$, $R = \sum_{i=1}^{n} y_i \overline{M}_i$.

As we illustrate in Section 6, the values of $x_1, \cdots x_m, y_1, \cdots, y_n, z$ as rational functions of $k_1, k_2, \cdots, k_m, \overline{k}_1, \overline{k}_2, \cdots, \overline{k}_n$ can be computed adhoc from any specific situation that we face in practice. We observe that Q is the weighted centroid of the complex points $M_1, M_2, \cdots, M_m$ using the weights $x_1, x_2, \cdots, x_m$ and R is the weighted centroid of the complex points $\overline{M}_1, \overline{M}_2, \cdots, \overline{M}_n$ using the weights $y_1, y_2, \cdots, y_n$. Of course, Theorem 1 is completely standard.

4 Complex Homothecy

If A, B are points in the complex plane, we denote $AB = B - A$. This also means that AB is the complex vector from A to B. Also, we define $|AB|$ to be the length of this vector AB. If k is any complex number, then $k = r(\cos\theta + i\sin\theta), r \geq 0$, is the polar form of k. It is assumed that the reader knows that $[r(\cos\theta + i\sin\theta)] \cdot [\overline{r}(\cos\phi + i\sin\phi)] = r \cdot \overline{r}(\cos(\theta + \phi) + i\sin(\theta + \phi))$.

Suppose $S, P, \overline{P}$ where $S \neq P, S \neq \overline{P}$ are points in the complex plane and $k = r(\cos\theta + i\sin\theta)$, $r > 0$, is a non-zero complex number. Also, suppose $S\overline{P} = k(SP)$ whereas always $S\overline{P} = \overline{P} - S$ and $SP = P - S$. Since $S\overline{P} = k(SP) = [r(\cos\theta + i\sin\theta)] \cdot (SP) = (\cos\theta + i\sin\theta) \cdot [r \cdot (SP)]$, we see that the complex vector $S\overline{P}$ can be constructed from the complex vector SP in the following two steps.

First, we multiply the vector SP by the positive real number (or scale factor) r to define a new vector, $SP' = r \cdot (SP)$. Since $SP' = P' - S$, the new point P' is colinear with S and

P with P, P' lying on the same side of S and $|SP'| = r \cdot |SP|$.

Next, we rotate the vector SP' by θ radians counterclockwise about the origin O as the axis to define the final vector $S\overline{P}$. Of course, the final point $\overline{P}$ itself is computed by rotating the point P' by θ radians counterclockwise about the axis S. If A, B, C, x, y are complex and $xA + yB = C, x + y = 1$, then $A + y(B - A) = C$. Therefore, $AC = y \cdot AB$ and if $y = r(\cos\theta + i\sin\theta), r \geq 0$, we see how to construct the point C.

From this construction, the following is obvious. Suppose $S \neq P$ are arbitrary variable points in the complex plane and $S\overline{P} = k \cdot (SP)$ where $k \neq 0$ is a fixed complex number.

Then the triangles $\triangle SP\overline{P}$ will always have the same geometric shape (up to similarity) since $\angle PS\overline{P} = \theta$ and $\left|S\overline{P}\right| : |SP| = r : 1$ when $k = r(\cos\theta + i\sin\theta), r > 0$. Next, let us suppose that the complex triangles $\triangle ABC$ and $\triangle\overline{ABC}$ and the complex point S are related as follows. $S\overline{A} = k \cdot (SA), S\overline{B} = k \cdot (SB), S\overline{C} = k \cdot (SC)$ where $k \neq 0$ is some fixed complex number.

We call this relation complex homothecy (or complex similitude). Also, S is the center of homothecy (or similitude) and k is the homothetic ratio (or ratio of similitude). When k is real we have the usual homothecy of two triangle. Of course, for both real or complex k, it is fairly obvious that $\triangle ABC$, and $\triangle\overline{ABC}$ are always geometrically similar and $\frac{|\overline{AB}|}{|AB|} = \frac{|\overline{AC}|}{|AC|} = \frac{|\overline{BC}|}{|BC|} = |k|$.

Of course, this same definition of complex homothecy also holds for two polygons $ABCDE, \cdots$ and $\overline{A}\ \overline{B}\ \overline{C}\ \overline{D}\ \overline{E}, \ldots$.

5 Using Theorem 1 to Create Endless Homothetic Triangles

Let $M_1, M_2, \cdots, M_m, \overline{M}_{a1}, \overline{M}_{a2}, \cdots, \overline{M}_{an}, \overline{M}_{b1}, \overline{M}_{b2}, \cdots, \overline{M}_{bn}, \overline{M}_{c1}, \overline{M}_{c2}, \cdots, \overline{M}_{cn}$ be any points in the plane.

As a specific example of this, we could start with a triangle $\triangle ABC$ and let $M_1, M_2, \cdots, M_m$ be any fixed points in the plane of $\triangle ABC$ such as the centroid, orthocenter, Lemoine point, incenter, Nagel point, etc.

Also, $\overline{M}_{a1}, \cdots, \overline{M}_{an}$ are fixed points that have some relation to side BC. $\overline{M}_{b1}, \cdots, \overline{M}_{bn}$ are fixed points that have some relation to side AC and $\overline{M}_{c1}, \cdots, \overline{M}_{cn}$ are fixed points that have some relation to side AB.

Let $k_1, k_2, \cdots, k_m, \overline{k}_1, \overline{k}_2, \cdots, \overline{k}_n$ be arbitrary but fixed complex numbers that satisfy $\sum\limits_{i=1}^{m} k_i + \sum\limits_{i=1}^{n} \overline{k}_i = 1$.

Define points P_a, P_b, P_c as follows.

1. $P_a = \sum\limits_{i=1}^{m} k_i M_i + \sum\limits_{i=1}^{n} \overline{k}_i \overline{M}_{ai}$.

2. $P_b = \sum\limits_{i=1}^{m} k_i M_i + \sum\limits_{i=1}^{n} \overline{k}_i \overline{M}_{bi}$.

3. $P_c = \sum\limits_{i=1}^{m} k_i M_i + \sum\limits_{i=1}^{n} \overline{k}_i \overline{M}_{ci}$.

Note that these points P_a, P_b, P_c are being defined in an analogous way. From Theorem 1, there exists complex numbers $x_1, x_2, \cdots, x_m$ where $\sum\limits_{i=1}^{m} x_i = 1$ and there exists complex numbers $y_1, y_2, \cdots, y_n$ where $\sum\limits_{i=1}^{n} y_i = 1$ and there exists a complex number z such that the following is true.

1. $x_1, \cdots, x_m, y_1, y_2, \cdots, y_n, z$ are rational functions of $k_1, \cdots, k_m \overline{k}_1, \cdots, \overline{k}_n$.

2. $P_a = Q + z\left(R_a - Q\right)$,

   $P_b = Q + z\left(R_b - Q\right)$,

   $R_c = P + z\left(R_c - Q\right)$ where

   $Q = \sum\limits_{i=1}^{m} x_i M_i$,

   $R_a = \sum\limits_{i=1}^{n} y_i \overline{M}_{ai}$,

   $R_b = \sum\limits_{i=1}^{n} y_i \overline{M}_{bi}$,

   $R_c = \sum\limits_{i=1}^{n} y_i \overline{M}_{ci},$.

   Equation 2 implies that Equation 3 is true since for example $P_a - Q = QP_a$.

3. $QP_a = z \cdot (QR_a)$,

   $QP_b = z \cdot (QR_b)$,

   $QP_c = z \cdot (QR_c)$.

From Section 4, Equation 3 implies that $\triangle P_aP_bP_c$ is homothetic to $\triangle R_aR_bR_c$ with a center of homothecy Q and a ratio of homothecy $\frac{QP_a}{QR_a} = \frac{QP_b}{QR_b} = \frac{QP_c}{QR_c} = z$. Also, of course, $\triangle P_aP_bP_c \sim \triangle R_aR_bR_c$ with a ratio of similarity $\frac{|P_aP_b|}{|R_aR_b|} = \frac{|P_aP_c|}{|R_aR_c|} = \frac{|P_bP_c|}{|R_bR_c|} = |z|$.

In the above construction, we could lump some (but not all) of the points $\{M_1, M_2, \cdots, M_m\}$ with each of the three sets of points $\left\{\overline{M}_{a1}, \cdots, \overline{M}_{an}\right\}, \left\{\overline{M}_{b1}, \cdots, \overline{M}_{bn}\right\}, \left\{\overline{M}_{c1}, \cdots, \overline{M}_{cn}\right\}$.

For example, we could deal with the four sets $\{M_2, \cdots, M_m\}, \left\{M_1, \overline{M}_{a1}, \cdots, \overline{M}_{an}\right\}$, $\left\{M_1, \overline{M}_{b1}, \cdots, \overline{M}_{bn}\right\}, \left\{M_1, \overline{M}_{c1}, \cdots, \overline{M}_{cn}\right\}$. We then use the same formulas as above and we have $QP_a = z \cdot (QR_a), QP_b = z \cdot (QR_b), QP_c = z \cdot (QR_c)$ where now $Q = \sum_{i=2}^{m} x_iM_i$, $R_a = \left(\sum_{i=1}^{n} y_i\overline{M}_{ai}\right) + y_{n+1}M_1$, $R_b = \left(\sum_{i=1}^{n} y_i\overline{M}_{bi}\right) + y_{n+1}M_1$, $R_c = \left(\sum_{i=1}^{n} y_i\overline{M}_{ci}\right) + y_{n+1}M_1$ where $\sum_{i=2}^{m} x_i = 1$, $\sum_{i=1}^{n+1} y_i = 1$.

As we illustrate in Section 7, by redefining our four sets $\{M_i\}, \left\{\overline{M}_{ai}\right\}, \left\{\overline{M}_{bi}\right\}, \left\{\overline{M}_{ci}\right\}$ in different ways, we can vastly expand our collections of homothetic triangles.

6 Two Specific Examples

Problem 1

Suppose $\triangle ABC$ lies in the complex plane. In $\triangle ABC$ let AD, BE, CF be the altitudes to sides BC, AC, AB respectively, where the points D, E, F lie on sides BC, AC, AB. The $\triangle DEF$ is called the orthic triangle of $\triangle ABC$. The three altitudes AD, BE, CF always intersect at a common point H which is called the orthocenter of $\triangle ABC$. Also, let O be the circumcenter of $\triangle ABC$ and let A', B', C' denote the midpoints of sides BC, AC, AB respectively. The line HO is called the Euler line of $\triangle ABC$. Define the points P_a, P_b, P_c as follows where k, e, m, n, r are fixed real numbers.

1. $AP_a = k \cdot AH + e \cdot HD + m \cdot AO + n \cdot AA' + r \cdot OA'$.

2. $BP_b = k \cdot BH + e \cdot HE + m \cdot BO + n \cdot BB' + r \cdot OB'$.

3. $CP_c = k \cdot CH + e \cdot HF + m \cdot CO + n \cdot CC' + r \cdot OC'$.

Show that there exists a point Q on the Euler line HO of $\triangle ABC$ and there exists a point R_a on side BC, a point R_b on side AC, a point R_c on side AB and there exists a real number z such that $\triangle P_aP_bP_c$ and $\triangle R_aR_bR_c$ are homothetic with center of homothecy Q and real ratio of homothecy $\frac{QP_a}{QR_a} = \frac{QP_b}{QR_b} = \frac{QP_c}{QR_c} = z$.

We can also show that there exists a point S on the Euler line OH such that this $\triangle R_aR_bR_c$ is the pedal triangle of S for $\triangle ABC$ where the pedal triangle is formed by the feet of the three perpendiculars from S to sides BC, AC, BC.

Solution We first deal with equation (1) given in Problem 1. Equations (2), (3) give analogous results.

Since $AP_a = P_a - A, AH = H - A, HD = D - A$, etc, we see that equation (1) is equivalent to $P_a - A = k\,(H - A) + e\,(D - H) + m\,(O - A) + n\,(A' - A) + r\,(A' - O)$. This is equivalent to $(**)$.

$(**)\; P_a = (1 - k - m - n)A + (k - e)\,H + eD + (m - r)\,O + (n + r)\,A'$.

From geometry, we know that $AH = 2 \cdot OA', BH = 2 \cdot OB', CH = 2 \cdot OC'$. Thus, $H - A = 2\,(A' - O)$ and $A = H + 2\,(O - A')$.

Substituting this value for A in $n\,(**)$ we have $P_a = (1 - k - m - n)\,(H + 2O - 2A') + (k - e)\,H + eD + (m - r)\,O + (n + r)\,A'$.

This is equivalent to the following.

$P_a = (1 - m - n - e)\,H + (2 - 2k - m - 2n - r)\,O + eD + (-2 + 2k + 2m + 3n + r)\,A'$.

Calling $1 - m - n - e = \theta, 2 - 2k - m - 2n - r = \phi, e = \lambda, -2 + 2k + 2m + 3n + r = \psi$, we have $P_a = \theta H + \phi O + \lambda D + \psi A'$ where $\theta + \phi + \lambda + \psi = 1$.

As in Theorem 1, we now lump H, O together and lump D, A' together.

Therefore, $P_a = [\theta H + \phi O] + [\lambda D + \psi A'] = (\theta + \phi)\left[\frac{\theta H}{\theta+\phi} + \frac{\phi O}{\theta+\phi}\right] + (\lambda + \psi)\left[\frac{\lambda D}{\lambda+\psi} + \frac{\psi A'}{\lambda+\psi}\right]$.

Calling $\frac{\theta H}{\theta+\phi} + \frac{\phi O}{\theta+\phi} = Q, \frac{\lambda D}{\lambda+\psi} + \frac{\psi A'}{\lambda+\psi} = R_a$, we have $P_a = (\theta + \phi)\,Q + (\lambda + \psi)\,R_a = Q + (\lambda + \psi)\,(R_a - Q) = Q + z\,(R_a - Q)$ where $z = \lambda + \psi = -2 + 2k + 2m + 3n + r + e$.

Of course, Q lies on the Euler line HO and R_a lies on the side BC since $\theta, \phi, \lambda, \psi$ are real.

By symmetry, equations (2), (3) yield the following analogous results.

$P_b = Q + z(R_b - Q)$ and $P_c = Q + z(P_c - Q)$ where $R_b = \frac{\lambda E}{\lambda+\psi} + \frac{\psi B'}{\lambda+\psi}, R_c = \frac{\lambda F}{\lambda+\psi} + \frac{\psi C'}{\lambda+\psi}$.

Of course, Q lies on the Euler line HO, R_a lies on side BC, R_b lies on side AC and R_c lies on side AB.

Since $QP_a = (\lambda + \psi)(QR_a) = z \cdot QR_a$,

$QP_b = (\lambda + \psi)(QR_b) = z \cdot QR_b$,

$QP_c = (\lambda + \psi)(QR_c) = z \cdot QR_c$,

we see that $\triangle R_aR_bR_c \sim \triangle P_aP_bP_c$ are homothetic with ratio of homothecy $\frac{QP_a}{QR_a} = \frac{QP_b}{QR_b} = \frac{QP_c}{QR_c} = z$.

Also, $\triangle R_aR_bR_c \sim \triangle P_aP_bP_c$ with ratio of similarity $\frac{|P_aP_b|}{|R_aR_b|} = \frac{|P_aP_c|}{|R_aR_c|} = \frac{|P_bP_c|}{|R_bR_c|} = |z|$.

Since D, E, F lie at the feet of the perpendiculars HD, HE, HF and since A', B', C' lie at the feet of the perpendiculars OA', OB', OC', it is easy to see that there exists a point S on the Euler line HO such that $\triangle R_aR_bR_c$ is the pedal triangle of S with respect to $\triangle ABC$.

We now deal with a special case of Problem 1. In Problem 1, let $k = e, m = n = r = 0$. Then $\theta = 1-e = 1-k, \phi = 2-2k, \lambda = k, \psi = -2+2k$. Also, $\theta+\phi = 3-3k, \lambda+\psi = -2+3k$.

Therefore, $Q = \frac{\theta H}{\theta+\phi} + \frac{\phi O}{\theta+\phi} = \frac{1}{3}H + \frac{2}{3}O$.

From geometry, we see that the center of homothecy is $Q = G$ where G is the centroid of $\triangle ABC$. Also, G is still the center of homothecy of $\triangle P_aP_bP_c$ and $\triangle R_aR_bR_c$ even for the case where k is complex.

Also, we see that $R_a = \frac{kD}{-2+3k} + \frac{(-2+2k)A'}{-2+3k}$.

Also, the ratio of homothecy is $z = -2 + 3k$.

If we let $k = e = 2, m = n = r = 0$, we see that $R_a = \frac{1}{2}D + \frac{1}{2}A', R_b = \frac{1}{2}E + \frac{1}{2}B', R_c = \frac{1}{2}F + \frac{1}{2}C'$.

From geometry we know that the nine point center N of $\triangle ABC$ lies at the mid point of the line segment HO.

Therefore, if $k = e = 2, m = n = r = 0$, we see that $\triangle R_aR_bR_c$ is the pedal triangle of the nine point center N. Also, when $k = e = 2, m = n = r = 0$, we see that $\triangle P_aP_bP_c$ is geometrically just the (mirror) reflections of vertices A, B, C about the three sides BC, AC, AB respectively. Also, the ratio of homothecy z is $z = -2 + 3k = 4$. Thus, $\triangle P_aP_bP_c$ is four times bigger than $\triangle R_aR_bR_c$.

Problem 2

Suppose $\triangle ABC$ lies in the complex plane. As in Problem 1, let AD, BE, CF be the altitudes for sides BC, AC, AB respectively where D, E, F lie on sides AB, AC, BC. Let I be the incenter of $\triangle ABC$ and let the incircle (I, r) be tangent to the sides BC, AC, AB at the points X, Y, Z respectively.

Define the points P_a, P_b, P_c as follows.

1. $P_a = D + i\,(IX)\,,$
2. $P_b = E + i\,(IY)\,,$
3. $P_c = F + i\,(IZ)$ where i is the unit imaginary.

We wish to find $\triangle R_aR_bR_c$ and a complex number z such that $\triangle P_aP_bP_c$ and $\triangle R_aR_bR_c$ are homothetic with a center of homothecy I and a complex ratio of homothecy $z = \frac{IP_a}{IR_a} = \frac{IP_b}{IR_b} = \frac{IP_b}{IR_b}$.

Solution

We first study what $\triangle P_aP_bP_c$ is geometrically. First, we note that $i \cdot IX, i \cdot IY, i \cdot IZ$ simply rotates the vectors IX, IY, IZ by $90°$ in the counterclockwise direction about the origin O as the axis. Also, we note that $|IX| = |X - I| = |IY| = |Y - I| = |IZ| = |Z - I| = r$ where r is the radius of the inscribed circle (I, r).

Therefore, the points P_a, P_b, P_c lie on sides BC, AC, AB respectively and the distance from D to P_a is r (going in the counterclockwise direction), the distance from E to P_b is r (going counterclockwise) and the distance from F to P_c is r (going counterclockwise).

We next analyze equation (1) in the problem. The analysis of equations (2), (3) is analogous.

Now equation (1) is equivalent to $P_a = D + i\,(X - I) = -i \cdot I + [iX + D] = -i \cdot I + (1 + i)\left[\frac{iX}{1+i} + \frac{D}{1+i}\right]$.

Obverse that $-i + (1 + i) = 1$ and $\frac{i}{1+i} + \frac{1}{1+i} = 1$.

Define $R_a = \frac{iX}{1+i} + \frac{D}{1+i} = D + \frac{i}{1+i}\,(X - D) = D + \frac{i}{1+i}\,(DX)$ since $X - D = DX$.

Therefore, $DR_a = \frac{i}{1+i}\,(DX) = \left(\frac{1+i}{2}\right)(DX)$ since $R_a - D = DR_a$.

Also, $P_a = -iI + (1 + i)\,R_a = I + (1 + i)\,(R_a - I)$. Therefore, $IP_a = (1 + i)\,(IR_a)$ since $P_a - I = IP_a$ and $R_a - I = IR_a$.

Therefore, by symmetry, we have the following equations.

1. $DR_a = \left(\frac{1+i}{2}\right)(DX), ER_b = \left(\frac{1+i}{2}\right)(EY), FR_c = \left(\frac{1+i}{2}\right)(FZ).$

2. $IP_a = (1+i)(IR_a), IP_b = (1+i)(IR_b), IP_c = (1+i)(IR_c).$

Equation (1) tells us how to construct $\triangle R_aR_bR_c$ from the points $\{D, X\}, \{E, Y\}, \{F, Z\}$.

Also, $\triangle P_aP_bP_c$ and $\triangle R_aR_bR_c$ are homothetic with center of homothecy I and complex ratio of homothecy $z = 1 + i = \frac{IP_a}{IR_a} = \frac{IP_b}{IR_b} = \frac{IP_c}{IR_c}$.

Also, $\triangle P_aP_bP_c \sim \triangle R_aR_bR_c$ and $\frac{|IP_a|}{|IR_a|} = \frac{|IP_b|}{|IR_b|} = \frac{|IP_c|}{|IR_c|} = |1+i| = \sqrt{2}$. Also, $\frac{|P_aP_b|}{|R_aR_b|} = \frac{|P_aP_c|}{|R_aR_c|} = \frac{|P_bP_c|}{|R_bR_c|}$.

7 Discussion

For a deeper understanding of the many applications of Theorem 1, we invite the reader to consider the following alternative form of Problem 1.

Problem 1 (alternate form) The statement of the definitions P_a, P_b, P_c is the same as in Problem 1.

However, we now define A'', B'', C'' to be the (mirror) reflections of O about the sides BC, AC, AB respectively. Therefore, $OA'' = 2 \cdot OA', OB'' = 2 \cdot OB', OC'' = 2 \cdot OC'$. We now substitute A'', B'', C'' for $A', B'C'$ in the problem by using $A'' - O = 2(A' - O)$, etc. and ask the reader to solve the same problem when we deal with $A, B, C, H, D, E, F, O, A'', B'', C''$ instead of $A, B, C, H, D, E, F, O, A',\ B', C'$. Also, we show that R_a, R_b, R_c will lie on lines DA'', EB'', FC'' instead of lying on sides BC, AC, BC. The pedal triangle part of the problem is ignored. The center of homothecy Q will still lie on the Euler line HO. This illustrates the endless way that Theorem 1 can be used to create homothetic triangles (and polygons).

References

[1] Court, Nathan A., College Geometry, Barnes and Noble, Inc., New York, 1963.