Special continued fraction expansions of a rational number with rational links in mind

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joint work with **Yuanan Diao** and **Claus Ernst**

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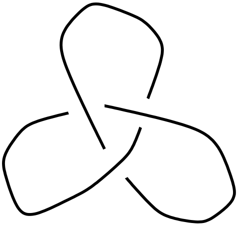

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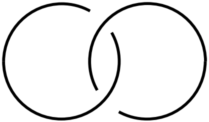

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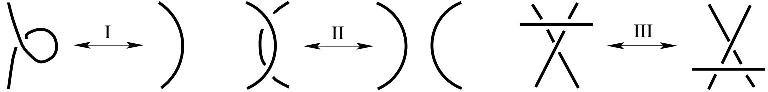

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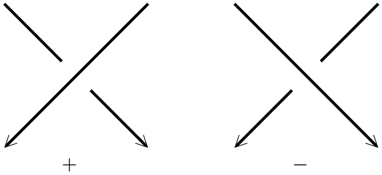

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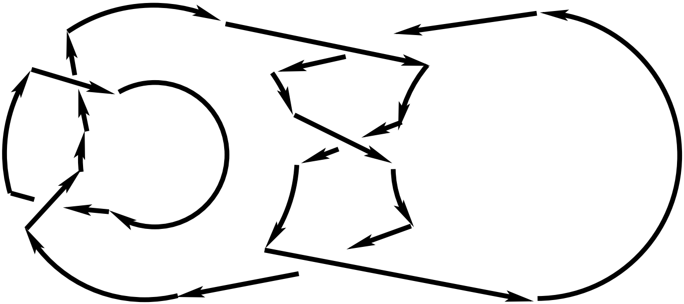

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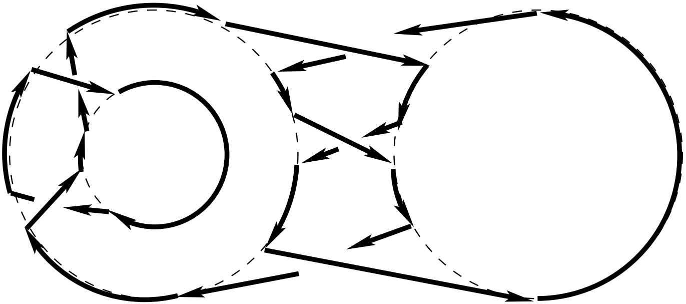

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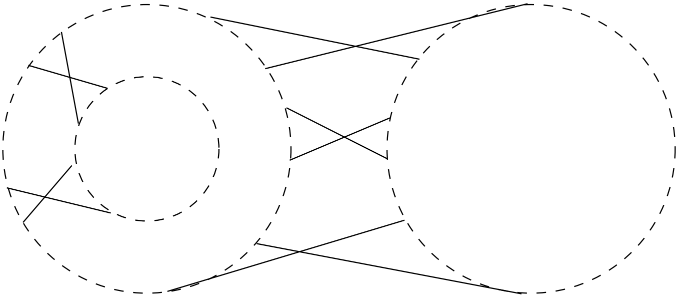

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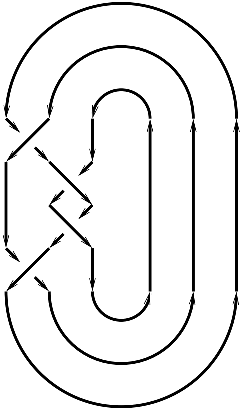

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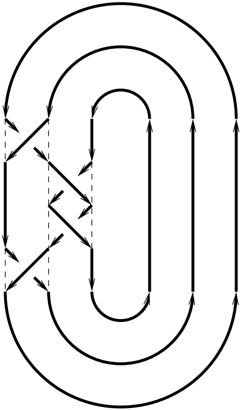

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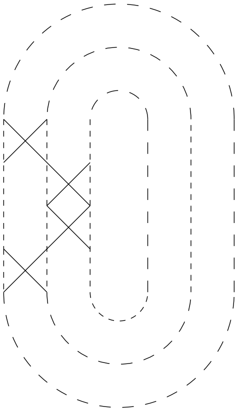

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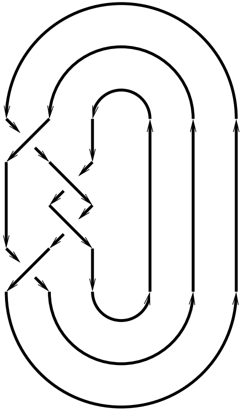

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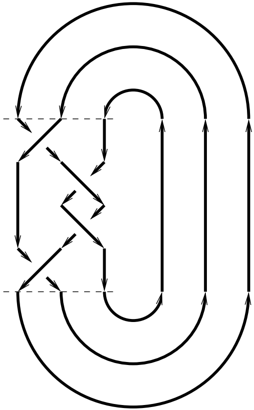

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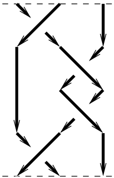

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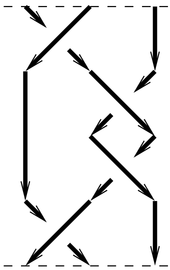

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The *braid index* is the least number of Seifert circles in the braid representation of an oriented link.

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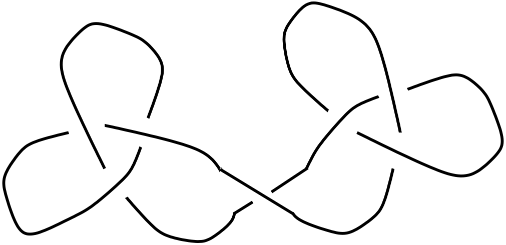

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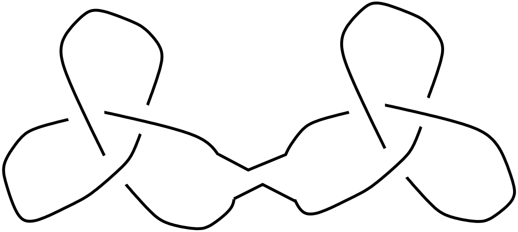

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Fact: If link has a reduced alternating diagram, then in this the number of crossings is minimal.

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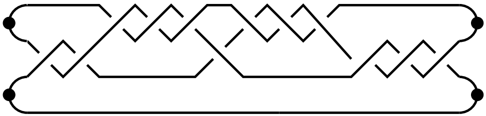

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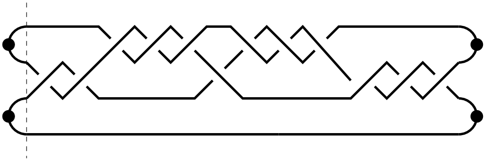

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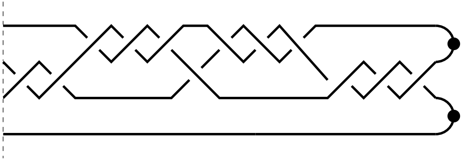

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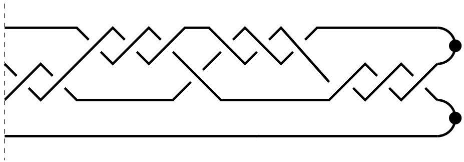

Cutting near the maxima we obtain a 2-*tangle.*

Finite simple continued fractions

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They are of the form

$$[c_0, \ldots, c_n] = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \ddots \cfrac{1}{c_{n-1} + \cfrac{1}{c_n}}}}$$

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where the *partial denominators* $c_0, \dots, c_n$ are integers and $c_n \neq 0$.

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Round towards the nearest even number (may fail at the end):

$$-\frac{9}{13} = [0, -2, 2, -5].$$

Rational links represented by continued fractions

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We encode an unoriented rational link diagram by $p/q = [0, a_1, a_2, \ldots, a_n]$ where $p/q \leq 1$ and satisfies $a_1 \cdots a_n \neq 0$, the numbers $|a_1|, \ldots, |a_n|$ are the numbers of consecutive half-turn twists in the twistboxes $B_1, \ldots, B_n$ following the sign convention below

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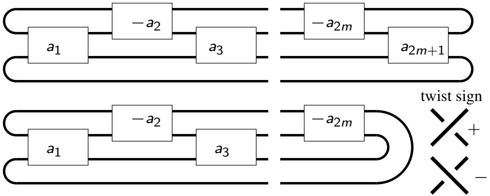

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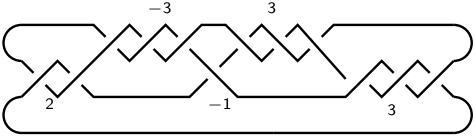

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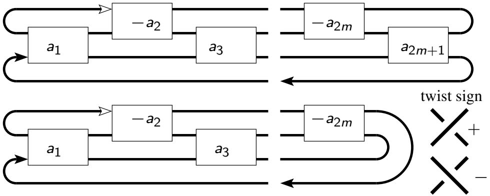

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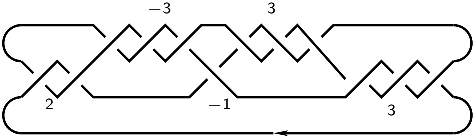

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Because every rational number has a continued fraction representation in which all partial denominators have the same sign.

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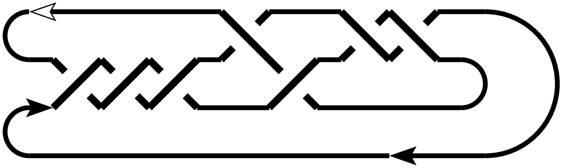

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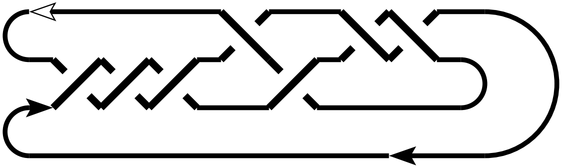

$5/18 = [0, 3, 1, 1, 2]$ (Independently of the orientation.)

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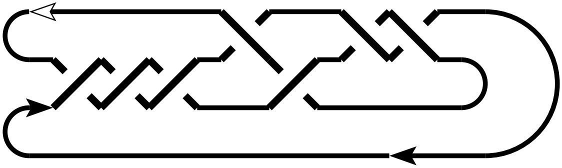
Fold up lowest strand:



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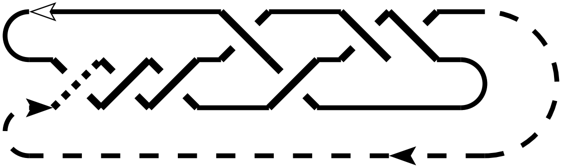

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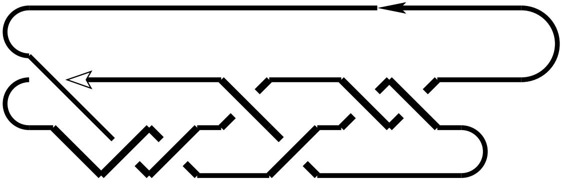

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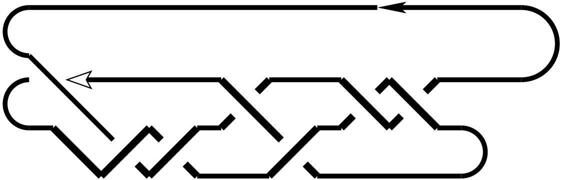
Reflect about a horizontal line:



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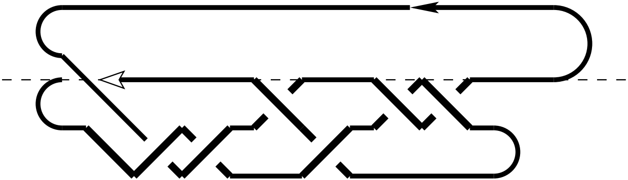

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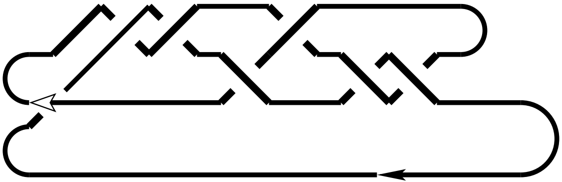

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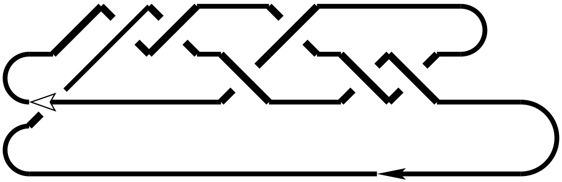

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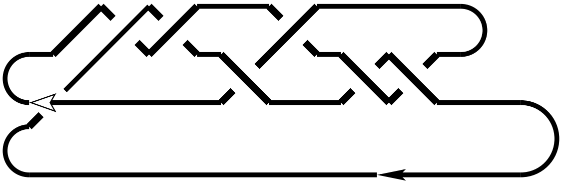

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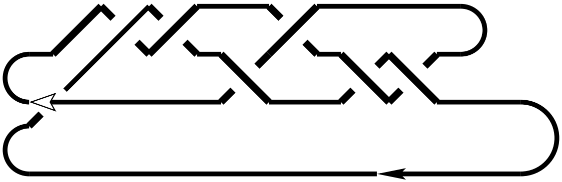

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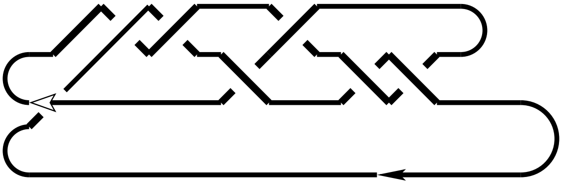

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Note also $[0, 3, 1, 1, 2] = [0, 3, 1, 1, 1, 1]$ since $a + 1/1 = a + 1$.

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Theorem (Murasugi)

Assume an oriented rational link is represented by $[2d_0, 2d_1, \dots, 2d_n]$*. Then the braid index of the link is* $\sum_{i=0}^{n} |d_i| - t + 1$ *where* t *is the number of indices* i *such that* $d_i d_{i+1} < 0$*.*

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Lemma

p/q *has a continued fraction expansion with only even partial denominators if and only if* pq *is even.*

Murasugi's braid index formula

Theorem (Murasugi)

Assume an oriented rational link is represented by $[2d_0, 2d_1, \ldots, 2d_n]$. Then the braid index of the link is $\sum_{i=0}^{n} |d_i| - t + 1$ where t is the number of indices i such that $d_i d_{i+1} < 0$.

Lemma

p/q has a continued fraction expansion with only even partial denominators if and only if pq is even.

If pq is odd then $q - p$ is even and $(q-p)/q = 1 - p/q$ encodes the mirror image of the link encoded by p/q.

Example and Remarks

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Issue: How to apply Murasugi's theorem to *alternating* rational links (where signs *don't* alternate)?

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for $p/q = [c_0, \ldots, c_n]$. We may think of continued fractions as transformations of the projective line, we may even write $1/0 = \infty$.

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Proposition

For $\delta \in \{-1, 1\}$, we may replace $[\dots, c_i, c_{i+1}, c_{i+2}, \dots, c_j, \dots, c_n]$ with $[\dots, c_i + \delta, -\delta, \delta - c_{i+1}, -c_{i+2}, \dots, -c_j, \dots, -c_n]$.

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Proof:

$$M(c_i)M(c_{i+1}) \begin{pmatrix} p \\ q \end{pmatrix} = M(c_i + \delta)M(-\delta)M(\delta - c_{i+1}) \begin{pmatrix} \delta p \\ -\delta q \end{pmatrix}$$

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In the last step we used $[\dots,u,0,v,\dots] = [\dots,u+v,\dots]$.

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We may increase the absolute value of any odd c_i by one, and replace c_{i+1} with $|c_{i+1}|-1$ copies of ± 2, and increase the absolute value of c_{i+2} by 1.

Transforming primitive blocks

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Theorem

Every $p/q \neq 0$ may be written as a finite simple continued fraction in a nonalternating form in two ways. Exactly one of these has a primitive block decomposition, which exceptional trivial primitive block if and only if pq is even.

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An exceptional trivial primitive block is a single odd partial denominator at the right end.

An automaton parsing the primitive blocks

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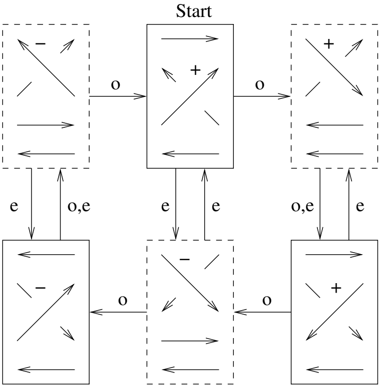

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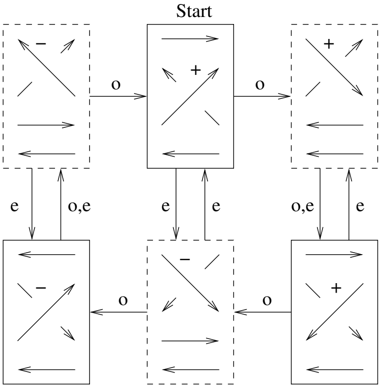

Miracle: The crossing sign $\varepsilon(a_i)$ changes exactly when we move to the next block.

A braid index formula

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Theorem

Suppose pq is even, and let $p/q = [c_0, \ldots, c_n]$ be the unique nonalternating continued fraction expansion that has a primitive block decomposition with $[c_{m_i}, c_{m_i+1}, \ldots, c_{m_i+2k_i}]$, $1 \leq i \leq \ell$ being the primitive blocks. Then the braid index associated to p/q may be computed by the following formula

$$1 + \sum_{1 \leq i \leq \ell} \sum_{0 \leq j \leq k_i} |c_{m_i+2j}|/2.$$

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For example, the braid index associated to $1402/1813 = [0, |1, 3, 2, 2, 3, |5, 1, 3]$ is $1 + (1 + 2 + 3)/2 + (5 + 3)/2 = 8$.

The Lickorish-Millet formula

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Proposition (Lickorish–Millett)

Let K be a rational knot or link, represented by the continued fraction $[0, c_1, \ldots, c_n]$ where the c_i are even integers. Then the HOMFLY polynomial $\mathcal{P}(K)$ is given by

$$\mathcal{P}(K)=\begin{pmatrix}1 & 0\end{pmatrix} \mathcal{M}((-1)^n c_n)\mathcal{M}((-1)^{n-1} c_{n-1}) \cdots \mathcal{M}(c_2)\mathcal{M}(-c_1)\begin{pmatrix} 1 \\ \frac{a^2-1}{az} \end{pmatrix}.$$

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Theorem

Suppose a rational link is represented by a nonalternating continued fraction $p/q = [0, a_1, \ldots, a_n]$ that has a primitive block decomposition with no exceptional primitive block. Then the HOMFLY polynomial may be written in matrix form as follows:

$$\mathcal{P}(K) = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathcal{H}(a_n)\,\mathcal{H}(a_{n-1}) \cdots \mathcal{H}(a_1) \begin{pmatrix} 1 \\ \frac{a^2-1}{az} \end{pmatrix}.$$

Here, after introducing $s = \text{sign}(a_1)$, and the *Fibonacci polynomials* $F_n(x)$ defined by $F_0(x) = 0$, $F_1(x) = 1$ and $F_{n+1}(x) = xF_n(x) + F_{n-1}(x)$, the matrices $\mathcal{H}(a_1), \mathcal{H}(a_2), \ldots, \mathcal{H}(a_n)$ are given by the following formulas.

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- $$\mathcal{H}(a_1) = \begin{cases} \mathcal{M}(-a_1) & \text{if } a_1 \text{ is even;} \\ \mathcal{M}(-(a_1 + s)) & \text{if } a_1 \text{ is odd.} \end{cases}$$

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- If $\varepsilon(a_i) \neq \varepsilon(a_{i-1})$ then set

$$\mathcal{H}(a_i) = \begin{cases} \mathcal{M}(-\varepsilon(a_i)a_i) & \text{if } a_i \text{ is even;} \\ \mathcal{M}(-\varepsilon(a_i)(a_i + s)) & \text{if } a_i \text{ is odd.} \end{cases}$$

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Here, after introducing $s = \text{sign}(a_1)$, and the *Fibonacci polynomials* $F_n(x)$ defined by $F_0(x) = 0$, $F_1(x) = 1$ and $F_{n+1}(x) = xF_n(x) + F_{n-1}(x)$, the matrices $\mathcal{H}(a_1), \mathcal{H}(a_2), \ldots, \mathcal{H}(a_n)$ are given by the following formulas.

- If $\varepsilon(a_i) = \varepsilon(a_{i-1}) = (-1)^i \cdot s$ then set $\mathcal{H}(a_i) =$

$$\begin{pmatrix} a^{\varepsilon(a_i)\cdot(a_i-s)} \cdot F_{|a_i|+1}(-\varepsilon(a_i)\cdot z) & a^{\varepsilon(a_i)\cdot a_i} \cdot F_{|a_i|}(-\varepsilon(a_i)\cdot z) \\ a^{\varepsilon(a_i)\cdot(a_i-2s)} \cdot F_{|a_i|}(-\varepsilon(a_i)\cdot z) & a^{\varepsilon(a_i)\cdot(a_i-s)} \cdot F_{|a_i|-1}(-\varepsilon(a_i)\cdot z) \end{pmatrix}.$$

THE END

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Epilogue: Yuanan Diao, Michael Finney and Dawn Ray recently proved that the number of oriented rational links with crossing number n and deficiency number d is

$$\Lambda_n(d) = F^{(d)}_{n-d-1} + \frac{1+(-1)^{nd}}{2} F^{(\lfloor d/2 \rfloor)}_{\lfloor n/2 \rfloor - \lfloor (d+1)/2 \rfloor}$$

where $F^{(d)}_n$ is the convolved Fibonacci sequence.

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Thank you very much!

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Thank you very much!

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