Sample Test II.

This study guide is subject to updates until Friday March 24.

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The real test will have approximately 5 questions plus 2 bonus questions and you will have about 50 minutes to answer them. The usage of books or notes, or communicating with other students will not be allowed. You will have to give the simplest possible answer and show all your work. Below I am only listing questions related to the definitions, theorems, and proofs I expect you to know. There will be also application exercises, similar to the already discussed homework questions.

1. Prove that the number of partitions of n into k parts is the same as the number of partitions of n with largest part of size k. (Hint: use Ferrers diagrams and transpose them. I will not give this hint on the test.)
2. Prove that the number of partitions of n into odd parts is the same as the number of partitions of n into self-conjugate partitions.
3. State the formula for the number of set partitions of n whose type vector is $1^{m_1}2^{m_2}\cdots n^{m_n}$. Explain the steps of the proof using the type vector $1^3 2^4 3^2$ as an example.
4. What kind of functions are counted by the partition number $P(n,k)$? (What is the size of the domain, the target, are the elements distinct or identical, are the functions one-to-one or onto?) (You may see a similar question regarding any entry in the table "The twelvefold way".)
5. If $p(n)$ is the number of all partitions of the number n, what is the number of partitions of n with no part of size 1?
6. Write the permutation
$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 8 & 4 & 1 & 9 & 6 & 5 & 2 & 7 \end{pmatrix}$$
as a product of disjoint cycles.
7. State and prove the inclusion-exclusion formula.
8. n persons attend a party. A fire breaks out in the building, while outside there is a heavy rain. Everybody rushes to the wardrobe, picks up an umbrella, and leaves. What is the probability that no one picked their own umbrella? Give an exact formula as a function of n and an approximate number for large n. Justify your answer.
9. State the Möbius inversion formula in the following two partially ordered sets: a chain with n elements, and the set of all positive divisors of n, ordered by the relation "divides".
10. Let $c(n,k)$ be the number of permutations of $[n]$ with k cycles. State an prove a recurrence for these numbers.

11. Prove that the number $c(n,k)$ in the previous question is the coefficient of x^k in $x(x+1)\cdots(x+n-1)$.
12. Define the Stirling numbers of the second kind, and use the answers to the previous two exercises to show that $s(n,k) = (-1)^{n-k}c(n,k)$.
13. Define the parity of a permutation and outline the proof of the fact that the parity is well defined.
14. What is the parity of a cycle of length k? Prove your statement. Use your result to describe the parity of a permutation in terms of its cycle decomposition.
15. What is the coefficient of x^n in $\dfrac{1}{(1-x)^m}$? Justify your answer!
16. Write $1 + x + x^2 + \cdots + x^n$ and $\sum_{n=0}^{\infty} x^n$ in closed form.
17. Find the coefficient of x^k in $\dfrac{2x}{(3+5x)^3}$.
18. Give a closed form formula for the sequence a_n given by $a_0 = 7$, $a_1 = 16$, and $a_{n+1} = 5a_n - 6a_{n-1}$. (Solution is $a_n = 5 \cdot 2^n + 2 \cdot 3^n$, but you have to show it.)
19. Using generating functions, prove that the sequence $a_0, a_1, \ldots$ given by the value of a_0 and the linear recurrence $a_n = \alpha \cdot a_{n-1} + \beta$ for $n \geq 1$ has the closed form formula

$$a_n = \begin{cases} a_0\alpha^n + \beta \cdot \frac{1-\alpha^n}{1-\alpha} & \text{if } \alpha \neq 1 \\ a_0\alpha^n + \beta \cdot n & \text{if } \alpha = 1 \end{cases}$$

20. Using generating functions, find a closed form formula for a_n when $a_0 = 2$ and $a_n = -a_{n-1} + 3$ for $n \geq 1$.

Good luck. Gábor Hetyei