Sample Test I.

The real test will have approximately 5 questions plus 2 bonus questions and you will have about 50 minutes to answer them. The usage of books or notes, or communicating with other students will not be allowed. You will have to give the simplest possible answer and show all your work. Below I am only listing questions related to the definitions, theorems, and proofs I expect you to know. There will be also application exercises, similar to the already discussed homework questions.

1. Prove by induction that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ holds for all $n \geq 1$.
2. State the pigeonhole principle.
3. Let $n = rm + 1$ and suppose we distribute n identical balls into m identical boxes. Prove that there is at least one box that receives at least $r + 1$ balls.
4. How many lists of length k can be formed using elements of the set $[n]$ if repetition of letters is not allowed? Justify your answer!
5. How many ways are there to line up 3 apples 4 oranges and 1 banana on a shelf? State the formula you are using and prove it!
6. How many lists of length k can be formed using elements of the set $[n]$ if repetition of letters is allowed? Justify your answer!
7. Explain how counting all subsets of $[n]$ is related to counting binary numbers.
8. State a formula for the number of k-element subsets of an n element set. Justify your answer by using your answer to question 4 and the equivalence principle.
9. Prove that the binomial coefficients satisfy $\binom{n}{k} = \binom{n}{n-k}$. Use the observation to calculate $\binom{100}{98}$.
10. State and prove Pascal's identity.
11. State a formula for the number of k-element multisets taken from an n-element set. Justify your answer by reducing your formula to your answer to question 8.
12. State and prove the binomial theorem.
13. State and prove the Chu-Vandermonde formula.
14. Use the multinomial theorem to find the coefficient of x^2y^3z in $(x + y + z)^6$. (Also state the theorem.)
15. State the general version of the binomial theorem and use it to find the coefficient of x^2 in $\sqrt[3]{1 - 2x}$.
16. State and prove the formula for the number of weak compositions of n into k parts.

17. Define the Stirling numbers of the second kind in terms of set partitions and prove that they satisfy the recurrence $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$.
18. Express the number of onto functions $f : [n] \to [k]$ in terms of the Stirling numbers of second kind, and explain the formula.
19. Outline the proof of the identity $x^n = \sum_{k=0}^{n} S(n,k) \cdot (x)_k$.
20. Define the Bell numbers $B(n)$ and prove that they satisfy the recurrence $B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i)$.
21. What is the number of all functions from a domain A of size k to a codomain B of size n if the elements of A and B are distinct? What is the number of $1-1$ functions and what is the number of onto functions under the same conditions? Justify your answer!
22. What is the number of all functions from a domain A of size k to a codomain B of size n if the elements of A are identical and the elements of B are distinct? What is the number of $1-1$ functions under the same conditions? Justify your answer!

Good luck. Gábor Hetyei