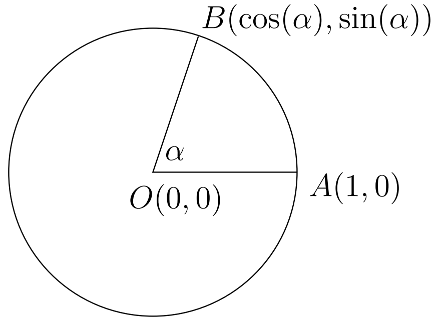
Trigonometric functions and complex numbers



The sine and cosine of an angle α is defined as the Cartesian coordinates of the point B on the unit circle centered at the oringin in the above picture.

The *Taylor series* of a function $f(x)$ at zero is defined as $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$. Here $f^{(n)}$ is the nth derivative of $f(x)$. For the sine and cosine functions, the Taylor series at zero converges to the function. The derivative of $\sin(x)$ is $\cos(x)$ and the derivative of $\cos(x)$ is $-\sin(x)$. We also know that $\cos(0) = 1$ and $\sin(0) = 0$. From these we obtain that

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{and}$$

$$\cos(x) = \frac{1}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Since the derivative of the exponential function e^x is itself, this function has the Taylor series expansion:

$$e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Combining these equations we get *Euler's formula*

$$e^{ix} = \cos(x) + i\sin(x),$$

where i is the square root of (-1).

Since $e^{i\alpha} \cdot e^{i\beta} = e^{i(\alpha+\beta)}$, using Euler's formula we get

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = (\cos(\alpha) + i\sin(\alpha))(\cos(\beta) + i\sin(\beta))$$

Expanding the right hand side, and comparing the real and imaginary parts we get

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \quad \text{and}$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta).$$

As a consequence of this approach, we also see that multiplying two unit complex numbers results in a unit complex number, whose *argument* is the sum of the arguments of the factor. The argument of a complex number is the angle between the vector representing it, and the positive real halfline. This observation may be easily generalized to arbitrary complex numbers. by Pythagoras' theorem, the length (called the *modulus*) of a complex number $a+bi$ is $||a+bi|| = \sqrt{a^2+b^2}$. Hence any nonzero complex number may be written as

$$a + bi = r(\cos(\alpha) + i\sin(\alpha)) = re^{i\alpha}$$

where $r = \sqrt{a^2 + b^2}$ is the modulus, and α is given by $\cos(\alpha) = a/r$ and $\sin(\alpha) = b/r$.

Using this form, we may multiply complex numbers as follows:

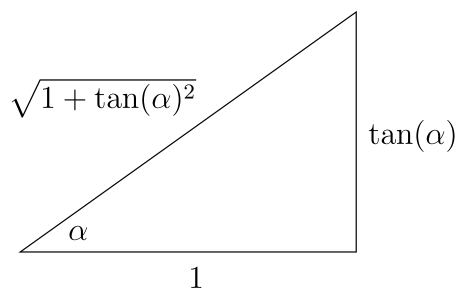
$$r_1 e^{i\alpha_1} \cdot r_2 e^{i\alpha_2} = r_1 r_2 e^{i(\alpha_1 + \alpha_2)}.$$

In other words, the modulus of the product is obtained by multiplying the moduli of the factors and the argument of the product is obtained by adding the arguments of the factors.

The tangent and cotangent functions may be defined as

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \text{and} \quad \cot(x) = \frac{\cos(x)}{\sin(x)}.$$

The following picture helps find the sine and cosine functions in terms of the tangent function.



Thus we have

$$\cos(\alpha) = \frac{1}{\sqrt{1+\tan(\alpha)^2}} \quad \text{and} \, \sin(\alpha) = \frac{\tan(\alpha)}{\sqrt{1+\tan(\alpha)^2}}.$$