Permutations, combinations, and variations

1 Permutations

Permutations are arrangements of objects (with or without repetition), *order does matter.*

The number of *permutations of n objects, without repetition*, is

$$P_n = P_n^n = n!.$$

The counting problem is the same as putting n distinct balls into n distinct boxes, or to count bijections from a set of n distinct elements to a set of n distinct elements.

A permutation with repetition is an arrangement of objects, where some objects are repeated a prescribed number of times. The number of *permutations with repetitions of k_1 copies of 1, k_2 copies of 2, ..., k_r copies of r* is

$$P_{k_1,\dots,k_r} = \frac{(k_1 + \cdots + k_r)!}{\prod_{i=1}^{r} k_i!}$$

The counting problem is the same as putting $k_1 + \cdots + k_r$ distinct balls into r distinct boxes such that box number i receives k_i balls. In other words we count onto functions from a set of $k_1 + \cdots + k_r$ distinct elements onto the set $\{1, 2, \dots, r\}$, such that the preimage of the element i has size k_i.

2 Combinations

Combinations are selections of objects, with or without repetition, *order does not matter.*

The number of *k-element combinations of n objects, without repetition* is

$$C_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The counting problem is the same as the number of ways of putting k identical balls into n distinct boxes, such that each box receives at most one ball. It is also the number of one-to-one functions from a set of k identical elements into a set of n distinct elements. It is also the number of k-element subsets of an n-element set.

The number of *k-element combinations of n objects, with repetition* is

$$\overline{C}_{n,k} = C_{n+k-1,k} = \binom{n+k-1}{k} = \left(\!\binom{n}{k}\!\right).$$

It is also the number of all ways to put k identical balls into n distinct boxes, or the number of all functions from a set of k identical elements to a set of n distinct elements.

3 Variations

Variations are arrangements of selections of objects, where *the order of the selected objects matters.* To count k-element variations of n objects, we first need to choose a k-element combination and then a permutation of the selected objects.

Thus the number of k-element variations of n elements with repetition not allowed is

$$V_{n,k} = P_{n,k} = \binom{n}{k} \cdot k! = (n)_k.$$

It is also the number of ways of putting k distinct balls into n distinct boxes such that each box receives at most one element. It is also the number of one-to-one functions from a set of k distinct elements into a set of n distinct elements.

The number of k-element variations of n-elements with repetition allowed, is

$$V_{n,k} = n^k.$$

It is the number of all ways of putting k distinct balls into n distinct boxes. It is also the number of all functions from a set of k distinct elements into a set of n distinct elements.