Assignment 13

Oral questions

1. Consider the fractional linear transformation $z \mapsto \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{R}$ and $ad-bc \neq 0$. Introduce $z = z_1+z_2 i$ and calculate explicitly the imaginary part of $\frac{az+b}{cz+d}$. Prove that the imaginary part of the image is positive for all $z_2 > 0$ if and only if $ad - bc > 0$.

   Now show that a conjugate fractional linear map $z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$ takes the upper half plane into itself if and only if $ad - bc < 0$.

2. Using $e^{-x} = \tan(\Pi(x)/2)$, prove the following formulas:

$$\sin(\Pi(x)) = \operatorname{sech}(x), \quad \cos(\Pi(x)) = \tanh(x), \quad \tan(\Pi(x)) = \operatorname{csch}(x).$$

Questions to be answered in writing

1. Prove that a fractional linear transformation that takes the Poincaré upper half plane onto itself may be written as $f(z) = \frac{az+b}{cz+d}$ where a, b, c, d are real numbers. (Hints: walk through the cases in the proof of Theorem 1 in the handout on fractional linear transformations. When c is not zero, you may assume it is a real number. You know that any fractional linear transformation may be written as a composition transformations that preserve half planes, except for a single inversion. That inversion should not take your half plane into the interior or the exterior of a circle.)

2. Find the Poincaré distance between the points $P = 3 + i$ and $Q = (6 + \sqrt{2})/2 + \sqrt{2}/2 \cdot i$ (in the Poincaré upper half plane model).