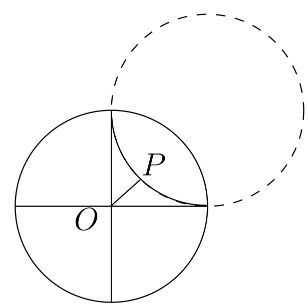
Assignment 11

Oral questions

1. Prove that the distance function $d(A, B) = |\log(AB, PQ)|$ of the Poincaré disk model is additive: if $A * C * B$ on a Poincaré line then $d(AC) + d(CB) = d(AB)$. Fix a Poincaré line with ideal points P and Q and a point A on it. Move another point B along the Poincaré line from P to Q. Show that $d(A, B)$ changes from ∞ to 0 and then back to ∞.

2. Schweikart's constant is the distance d for which the angle of parallelism is $\Pi(d) = 45^o$. Prove that for the length function of the Poincaré disk model, Schweikart's constant equals $\log(1 + \sqrt{2})$. You may use the following formula in your proof. If a point P is at a Euclidean distance r from the center O then its hyperbolic distance from O is

$$d(O, P) = \ln\left(\frac{1 + r}{1 - r}\right).$$



Questions to be answered in writing

1. Let a, b, c, d be real numbers, such that $ad - bc \neq 0$. Using that

$$\frac{az + b}{cz + d} = \begin{cases} \frac{a}{c} + \frac{b - ad/c}{cz + d} & \text{if } c \neq 0, \text{ and} \\ \frac{az + b}{d} & \text{if } c = 0, \end{cases}$$

show that every fractional linear transformation of the above form arises as a combination of horizontal translations $z \mapsto z + b$, dilations $z \mapsto az$ and "reflected inversions" $z \mapsto 1/z$. Conclude that fractional linear transformations preserve angles and the cross-ratio.

2. A hyperbolic circle centered at C of radius r is the set of all points A satisfying $d(A, C) = r$. Prove that a hyperbolic circle corresponds to a circle in the Poincaré disk model, although its center may not be equal to its Euclidean center. (Hint: prove the statement in the case when $C = P$ first, where P is the center of the Poincaré disk. Use then the fact that reflection about the perpendicular bisector of PC takes a hyperbolic circle centered at P into a hyperbolic circle centered at C, and that this reflection corresponds to an inversion about a circle.)