The sensed ratio

1 Definition and basic properties

Let A and B be two fixed points in the plane, and let P be any point on the line AB that is different from A and B. We define the *sensed ratio* of P with respect to the base AB as the quotient

$$(ABP) := \frac{AP}{PB}.$$

Here the lengths are signed lengths, in other words, we consider AP to be the negative of PA.

Lemma 1 *The sensed ratio (ABP) is positive if and only if P is between A and B. Otherwise (ABP) is negative.*

Indeed, the directed segments AP and PB have the same direction exactly when P is between A and B. We may refine the previous statement using the following lemma.

Lemma 2 *If a, b and c are positive then*

$$\frac{a}{b} < \frac{a+c}{b+c} \quad \textit{iff} \quad a < b \quad \textit{and}$$

$$\frac{a}{b} > \frac{a+c}{b+c} \quad \textit{iff} \quad a > b.$$

Proof: The statement follows from the fact that $a/b < (a+c)/(b+c)$ is equivalent to $a(b+c) < b(a+c)$, which is equivalent to $a < b$. $\Diamond$

We may rephrase Lemma 2 as follows: if we increase the numerator and the denominator of a positive fraction by the same number, then the quotient increases for fractions that are less than 1 and decreases for fractions that are more than 1. Note also that a/b compares to 1 the same way as $(a+c)/(b+c)$.

Theorem 1 *The value of (ABP) determines the position of the point P relative to A and B as follows:*

(i) $(ABP) > 0$ *if and only if P is between A and B;*
(ii) $(ABP) < -1$ *if and only if B is between A and P;*
(iii) $-1 < (ABP) < 0$ *if and only if A is between B and P.*

Furthermore, given any real number $r \notin \{-1, 0\}$, there is a unique point on the line AB satisfying $(ABP) = r$.

Proof: Statement (i) is the same as Lemma 1. Assume from now on that P is not between A and B. Observe that the absolute value of AP is less than the absolute value of PB exactly when A is between B and P, and the absolute value of AP is more than the absolute value of PB if and only if B is between A and P. This concludes the proof if (ii) and (iii).

To prove the last statement let us describe how (ABP) changes as P moves along the line AB.

1. If P is between A and B, the sensed ratio (APB) strictly increases as P moves from A towards B. The value of (ABP) gets infinitely close to zero as P moves towards A, and it gets arbitrarily large as P moves towards B. Thus (ABP) attains every positive real number exactly once.
2. If B is between A and P or, in other words, P is beyond B then, by Lemma 2, the absolute value of (ABP) strictly decreases as P moves away from B and strictly increases as P moves closer to B. As P moves arbitrarily close to B, the value of (APB) gets arbitrarily close to $-\infty$, as P moves arbitrarily far from B, the value of (APB) gets arbitrarily close to -1. Thus (ABP) attains every value in $(-\infty, -1)$ exactly once.
3. If A is is between B and P or, in other words, P is beyond A then, by Lemma 2, the absolute value of (ABP) strictly increases as P moves away from A and strictly decreases as P moves closer to A. As P moves arbitrarily close to A, the value of (APB) gets arbitrarily close to 0, as P moves arbitrarily far from A, the value of (APB) gets arbitrarily close to -1. Thus (ABP) attains every value in $(-1, 0)$ exactly once.

Summarizing the three cases we find that (ABP) attains every real number exactly once, except for 0 and -1. $\diamondsuit$

Note that we may think of $(ABP) = 0$ as the "limit position" when $P = A$, and of $(ABP) = -1$ as the "limit position" when P is "the point at infinity". For the degenerate case $P = B$ we would need to set $(ABP) = \pm\infty$ suggesting that we would need to replace the usual model of real numbers with a model where we close the number line with a single point at infinity.

2 Ceva's and Menelaus' theorems in terms of the sensed ratio

Let A, B, C be three points, not all on the same line. Let $A_1 \not\in \{B, C\}$ be a point on the line BC $B_1 \not\in \{A, C\}$ be a point on the line AC and $C_1 \not\in \{A, B\}$ be a point on the line AB. Ceva's theorem and Menelaus' theorem may be rephrased as follows.

Theorem 2 (Ceva) *The lines AA_1, BB_1 and CC_1 are concurrent if and only if*

$$(ABC_1)(BCA_1)(CAB_1) = 1.$$

Theorem 3 (Menelaus) *The points A_1, B_1 and C_1 are collinear if and only if*

$$(ABC_1)(BCA_1)(CAB_1) = -1.$$

For either theorem, we only need a geometric proof for the "only if" part, i.e. we only need to show that the assumed concurrence, respectively collinearity implies the equation for the sensed ratios. For these proofs we refer to Prof. Royster's lecture notes [3]. Proving the reverse implication becomes then very easy using Theorem 1. For example to prove the "if" part of Ceva's theorem, let P be the intersection of the lines AA_1 and BB_1 and let C_1^* be the intersection of CP and AB. By the "only if" part of Ceva's theorem, we get

$$(ABC_1^*)(BCA_1)(CAB_1) = 1.$$

Comparing this with the assumed equation

$$(ABC_1)(BCA_1)(CAB_1) = 1,$$

we get $(ABC_1^*) = (ABC_1)$. As a consequence of Theorem 1 the points C_1 and C_1^* must coincide proving that the lines AA_1, BB_1 and CC_1 are concurrent.

3 Affine transformations

Affine transformations [2, 4] are defined as continuous transformations of the plane that preserve lines and parallelism. Equivalently they preserve lines and ratios, or lines and sensed ratios. Every affine transformation is a (finite) composition of rotations, translations, dilations, and shears [2].

4 Projective transformations and the cross-ratio

Projective transformations or *homographies* [5] are continuous transformations of the real projective plane [6] that take lines into lines and preserve incidence. It is possible to show that they may be written as a (finite) composition of central and parallel projections. A projective transformation does not necessarily preserve the sensed ratio. In fact, we have the following result.

Theorem 4 *Let ℓ_1 and ℓ_2 any two lines. Let A_i, B_i and C_i be any three points on the line ℓ_i (where $i = 1, 2$). Then there is a projective transformation that takes A_1 into A_2, B_1 into B_2 and C_1 into C_2. This transformation arises as the composition of at most two (central or parallel) projections.*

Proof: Let ℓ_1' be the line parallel to ℓ_1 through A_2 (see Figure 1). The parallel projection from ℓ_1 to ℓ_1' takes A_1 into A_2. Let B_1', respectively C_1' the image of B_1, respectively C_1' under this parallel projection. Let P be the intersection of the line $B_1'B_2$ and $C_1'C_2$. (Here P is an ideal point if $B_1'B_2$ and $C_1'C_2$ are parallel.) The central (or parallel) projection from P that takes ℓ_1' into ℓ_2 leaves A_2 fixed, takes B_1' into B_2 and C_1' into C_2. $\Diamond$

On the other hand, it is possible to show that projective transformation preserve *cross-ratios.* Given four collinear points A, B, C, and D, the *cross-ratio* $(ABCD)$ is given by

$$(ABCD) = \frac{(ABC)}{(ABD)} = \frac{AC}{CB} \cdot \frac{DB}{AD}.$$

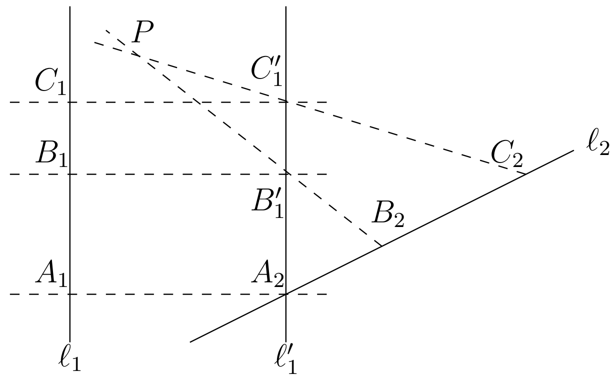

Figure 1: Transforming ℓ_1 into ℓ_2

Since every projective transformation is a composition of finitely many projections, one only needs to show that projections preserve the cross-ratio. This is obvious for parallel projections since they even preserve sensed ratios, and the cross-ratio is the quotient of two sensed ratios. Thus one only needs to show that central projections preserve the cross-ratio, which is not hard [1].

References

[1] Cut The Knot Entry: Cross-Ratio
`http://www.cut-the-knot.org/pythagoras/Cross-Ratio.shtml`

[2] Mathworld entry: Affine transformation
`http://mathworld.wolfram.com/AffineTransformation.html`

[3] D. Royster, "Non-Euclidean Geometry and a Little on How We Got There," Lecture notes, December 11, 2011.

[4] Wikipedia entry: Affine transformation
`http://en.wikipedia.org/wiki/Affine_transformation`

[5] Wikipedia entry: Homography
`http://en.wikipedia.org/wiki/Homography`

[6] Wikipedia entry: Real Projective Plane
`http://en.wikipedia.org/wiki/Real_projective_plane`