The spherical Pythagorean theorem

Proposition 1 *On a sphere of radius R, any right triangle $\triangle ABC$ with $\angle C$ being the right angle satisfies* $\cos(c/R) = \cos(a/R)\cos(b/R)$.

Proof: We complement the proof presented in [1, page 206]. Let O be the center of the sphere, we may assume its coordinates are $(0,0,0)$. We may rotate the sphere so that A has coordinates $\overrightarrow{OA} = (R,0,0)$ and C lies in the xy-plane. Rotating around the z axis by $\beta := \angle AOC$ takes A into C. The edge OA moves in the xy-plane, by β, thus the coordinates of C are $\overrightarrow{OC} = (R\cos(\beta), R\sin(\beta), 0)$. Since we have a right angle at C, the plane of $\triangle OBC$ is perpendicular to the plane of $\triangle OAC$ and it contains the z axis. An orthonormal basis of the plane of $\triangle OBC$ is given by $1/R \cdot \overrightarrow{OC} = (\cos(\beta), \sin(\beta), 0)$ and the vector $\overrightarrow{OZ} := (0,0,1)$. A rotation around O in this plane by $\alpha := \angle BOC$ takes C into B:

$$\overrightarrow{OB} = \cos(\alpha) \cdot \overrightarrow{OC} + \sin(\alpha) \cdot R \cdot \overrightarrow{OZ} = (R\cos(\beta)\cos(\alpha), R\sin(\beta)\cos(\alpha), \sin(\alpha)).$$

Introducing $\gamma := \angle AOB$, we have

$$\cos(\gamma) = \frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{R^2} = \frac{R^2\cos(\alpha)\cos(\beta)}{R^2}.$$

The statement now follows from $\alpha = a/R$, $\beta = b/R$ and $\gamma = c/R$. $\Diamond$

To prove the rest of the formulas of spherical trigonometry, we need to show the following.

Proposition 2 *Any spherical right triangle $\triangle ABC$ with $\angle C$ being the right angle satisfies*

$$\sin(A) = \frac{\sin\left(\frac{a}{R}\right)}{\sin\left(\frac{c}{R}\right)} \quad \textit{and} \tag{1}$$

$$\cos(A) = \frac{\tan\left(\frac{b}{R}\right)}{\tan\left(\frac{c}{R}\right)}. \tag{2}$$

Proof: We complement the proof presented in [1, page 208]. After replacing a/R, b/R and c/R with a, b, and c we may assume $R = 1$. This time we rotate the triangle in such a way that $\overrightarrow{OC} = (0,0,1)$, A is in the xz plane and B is in the yz-plane. A rotation around O in the xz plane by $b = \angle AOC$ takes C into A, thus we have

$$\overrightarrow{OA} = (\sin(b), 0, \cos(b)).$$

Similarly, a rotation around O in the yz-plane by $a = \angle BOC$ takes C into B, thus we have

$$\overrightarrow{OB} = (0, \sin(a), \cos(a)).$$

The angle A is between $\overrightarrow{OA} \times \overrightarrow{OB}$ and $\overrightarrow{OA} \times \overrightarrow{OC}$. Here

$$\overrightarrow{OA} \times \overrightarrow{OB} = (-\cos(b)\sin(a), -\sin(b)\cos(a), \sin(a)\sin(b)) \quad \text{and} \quad \overrightarrow{OA} \times \overrightarrow{OC} = (0, -\sin(b), 0).$$

The length of $\overrightarrow{OA} \times \overrightarrow{OB}$ is $\left|\overrightarrow{OA}\right| \cdot \left|\overrightarrow{OB}\right| \cdot \sin(c) = \sin(c)$, the length of $\overrightarrow{OA} \times \overrightarrow{OC}$ is $\sin(b)$.

To prove (1) we use the fact that

$$\left|(\overrightarrow{OA} \times \overrightarrow{OB}) \times (\overrightarrow{OA} \times \overrightarrow{OC})\right| = \left|\overrightarrow{OA} \times \overrightarrow{OB}\right| \cdot \left|\overrightarrow{OA} \times \overrightarrow{OC}\right| \cdot \sin(A). \tag{3}$$

Since

$$(-\cos(b)\sin(a), -\sin(b)\cos(a), \sin(a)\sin(b)) \times (0, -\sin(b), 0) = (\sin(a)\sin^2(b), 0, \sin(b)\cos(b)\sin(a))$$

the left hand side of (3) is

$$\sqrt{\sin^2(a)\sin^4(b) + \sin^2(b)\cos^2(b)\sin^2(a)} = \sin(b)\sin(a)\sqrt{\sin^2(b) + \cos^2(b)} = \sin(b)\sin(a).$$

The right hand side of (3) is is $\sin(b)\sin(c)\sin(A)$. Thus we have

$$\sin(b)\sin(a) = \sin(b)\sin(c)\sin(A),$$

yielding (1).

To prove (2) we use the fact that

$$(\overrightarrow{OA} \times \overrightarrow{OB}) \cdot (\overrightarrow{OA} \times \overrightarrow{OC}) = \left|\overrightarrow{OA} \times \overrightarrow{OB}\right| \cdot \left|\overrightarrow{OA} \times \overrightarrow{OC}\right| \cdot \cos(A). \tag{4}$$

The left hand side is $\sin^2(b)\cos(a)$, the right hand side is $\sin(b)\sin(c)\cos(A)$. Thus we obtain

$$\sin^2(b)\cos(a) = \sin(b)\sin(c)\cos(A),$$

yielding

$$\cos(A) = \frac{\sin(b)\cos(a)}{\sin(c)} = \frac{\tan(b)}{\tan(c)} \cdot \frac{\cos(a)\cos(b)}{\cos(c)}.$$

Equation (2) now follows from Proposition 1. $\diamondsuit$

References

[1] D. Royster, “Non-Euclidean Geometry and a Little on How We Got There,” Lecture notes, December 11, 2011.