The crossbar theorem

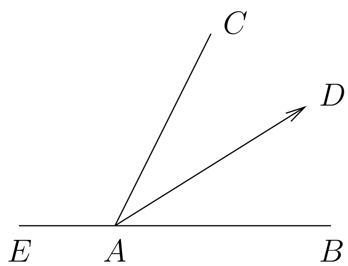
corrected proof of Theorem 7.10 in [1]

Proposition 1 *Assume D is in the interior of the angle $\angle BAC$. Then*

(i) *every point of the ray $\overrightarrow{AD}:$ except for A is also in the interior of the the angle $\angle BAC$;*

(ii) *no point of the ray opposite to $\overrightarrow{AD}:$ is in the interior;*

(iii) *If $B * A * E$ then C is in the interior of $\angle DAE$.*



Theorem 1 (Crossbar theorem) *Given $\triangle ABC$, let D be a point in the interior of $\angle BAC$. Then there is a point G so that G lies on both $\overrightarrow{AD}:$ and BC*

Proof: (Use illustration from [1, Theorem 7.10].) Let $\overrightarrow{AF}:$ be the opposite ray to $\overrightarrow{AD}:$. If $\overrightarrow{AF}: \cap BC = \{P\}$, then $B * P * C$ and, by [1, Theorem 7.7], we have that P lies in the interior of $\angle BAC$. However, this contradicts part (ii) of Proposition 1 above. Thus, we have that $\overrightarrow{AF}: \cap BC = \emptyset$. Now, this means that $\overleftrightarrow{AD} \cap BC = \emptyset$ since neither $\overrightarrow{AD}:$ nor its opposite ray intersect BC. It follows that B and C are on the same side of $\overleftrightarrow{AD}$.

Let E be a point on the line $\overleftrightarrow{AB}$ such that $B * A * E$. Then, by part (iii) of Proposition 1 above C is in the interior of $\angle DAE$. As a consequence, E and C are on the same side of $\overleftrightarrow{AD}$. Therefore E, B and C are all on the same side of $\overleftrightarrow{AD}$, in contradiction with $B * A * E$. $\Diamond$

References

[1] D. Royster, "Non-Euclidean Geometry and a Little on How We Got There," Lecture notes, December 11, 2011.