
Study Guide for Test 1.

The real test will have less questions and you will have about 80 minutes to answer them. The usage of books or notes, or communicating with other students will not be allowed. You will have to give the simplest possible answer and show all your work. Below you find sample questions and indications which theorems and proofs you will have to remember from the book. Review also all past homework questions as questions similar to them might appear on the test.

1. Prove that in a semigroup, every left inverse is equal to every right inverse.
2. Define a vectorspace. (No need to remember the axiom numbers, but you should remember all required properties).
3. Which of the following is a vectorspace? Justify your answer!
 - Polynomials in one variable with integer coefficients, rational scalars.
 - Polynomials in one variable with rational coefficients, of degree at most 5, rational scalars.
 - Polynomials in one variable with rational coefficients, of degree at least 5, rational scalars.
4. Prove the cancellation law for vector addition.
5. You have to be able to prove any part of Theorem 1.2. (I will state it for you.)
6. Define a subspace. Which part of the definition is superfluous? Justify your statement.
7. Consider \mathbb{R}^2 as the subset $\{(a_1, a_2) \in \mathbb{C}^2 : a_1, a_2 \in \mathbb{R}\}$ of \mathbb{C}^2 . Is this subset a subspace if the field of scalars is \mathbb{C} ? What if the field of scalars is \mathbb{R} ? Justify your answer.
8. Define the linear span of a set and prove that it is the smallest subspace containing the set.
9. Explain why the linear span, as defined in the book, is closed under addition, even when you have vectors showing up in both summands, e.g., when you add $v_1 + 2v_2$ to $(-v_1 + v_2)$.
10. Solve the following system of equations, using the Gauss-Jordan method:

$$\begin{aligned}x_1 - 2x_2 + x_4 &= 5 \\-x_1 + 2x_2 + x_3 &= 1 \\x_3 - x_4 &= 0\end{aligned}$$

11. Explain why the Gauss-Jordan method finds all solutions of a system of linear equations.
12. Define linear dependence and show an example of three linearly dependent vectors in \mathbb{R}^3 .

13. Prove that the subset of a linearly independent set is linearly independent (Theorem 1.6).
14. Let S be a linearly independent subset of a vector space and $v \notin S$ a vector. Prove that $\{v\} \cup S$ is linearly dependent if and only if $v \in \text{span}(S)$ (Theorem 1.7).
15. Define a basis.
16. Prove that every finite generating set contains a finite basis.
17. State and prove the Replacement Theorem (Theorem 1.10)
18. Use the replacement theorem to prove that every finite basis has the same number of vectors.
19. Find the dimension of the following vector spaces: polynomials of degree at most n , $m \times n$ matrices, symmetric $n \times n$ matrices.
20. Give an example of an infinite dimensional vector space.
21. Define a linear transformation, and verify that taking derivatives is a linear transformation on the vector space of polynomials.
22. Define the rank and nullity of a linear transformation.
23. Consider a linear transformation whose domain is finite dimensional. State and prove the formula connecting the rank and nullity of the linear transformation and the dimension of the domain.
24. Prove that for a linear transformation between finite dimensional vector spaces of the same dimension it is equivalent to have a one-to-one or an onto transformation.
25. Describe the matrix representation of a linear transformation.
26. Let $T : P_3(\mathbb{R}) \rightarrow P_4(\mathbb{R})$ given by $p(x) \mapsto (x - 2) \cdot p(x)$. Write down the matrix representation of this linear transformation with respect to the ordered bases $1, x, x^2, x^3$ and $1, x, x^2, x^3, x^4$.

Good luck.

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