The Möbius function of a partially ordered set

1 Partially ordered sets

A *partially ordered set (poset)* P is a set with a relation $\leq$ that is

- *Reflexive:* $x \leq x$ holds for all $x \in P$;
- *Antisymmetric:* if $x \leq y$ and $y \leq x$ then $x = y$;
- *Transitive:* if $x \leq y$ and $y \leq z$ then $x \leq z$.

Examples of partially ordered sets include: the subsets of a set, ordered by inclusion, the real numbers, ordered by the usual $\leq$ relation, and the natural numbers, ordered by the relation "divides".

2 Locally finite posets

An *interval* $[x, y]$ in a poset is the set of all elements z satisfying $x \leq z \leq y$. A partially ordered set is *locally finite* if each of its intervals has only finitely many elements.

Let $f : P \to \mathbb{R}$ be a function. The *lower sum* $S_{\leq} f$ and *upper sum* $S_{\geq} f$ are given by

$$(S_{\leq} f)(x) = \sum_{y \leq x} f(y) \quad \text{and} \quad (S_{\geq} f)(x) = \sum_{y \geq x} f(y).$$

The *Möbius function* μ of a locally finite poset assigns a number to each interval $[x, y] \subseteq P$ according to the following rules:

- $\mu(x, x) = 0$ holds for all $x \in P$
- For all $x < y$ we have $\sum_{x \leq z \leq y} \mu(x, z) = 0$.

This function is used in the famous *Möbius inversion formula*:

Theorem 2.1 *Let P be a locally finite poset, and f and g functions from P to $\mathbb{R}$. Then $g(x) = S_{\leq} f(x)$ if and only if $f(x) = \sum_{y \leq x} g(y)\mu(y, x)$ and $g(x) = S_{\geq} f(x)$ if and only if $f(x) = \sum_{y \geq x} g(y)\mu(x, y)$.*

3 Examples

1. If P is the set of all finites subsets of the set $\{1, 2, 3, \ldots\}$, ordered by inclusion, then the Möbius function is $\mu(X, Y) = (-1)^{|Y|-|X|}$. The Möbius inversion formula implies

$$g(X) = \sum_{Y \subseteq X} f(Y) \quad \text{iff} \quad f(X) = \sum_{Y \subseteq X} g(Y)(-1)^{|X|-|Y|}$$

and, for all subsets of $\{1, 2, \ldots, n\}$,

$$g(X) = \sum_{\{1,\ldots,n\} \supseteq Y \supseteq X} f(Y) \quad \text{iff} \quad f(X) = \sum_{\{1,\ldots,n\} \supseteq Y \supseteq X} g(Y)(-1)^{|Y|-|X|}.$$

Assume $A_1, A_2, \ldots, A_n$ are subsets of $\{1, 2, \ldots, N\}$. For any $Y \subseteq \{1, 2, \ldots, n\}$ let $f(Y)$ be the number of elements of $\bigcup_{y \in Y} A_y \cap \bigcap_{x \notin Y} \overline{A_x}$. (This is the number of elements belonging to all A_y for $y \in Y$ and no other A_x.) Let $g(X)$ be the number of elements in $\bigcap_{x \in X} A_x$. Then $g(X)$ is clearly equal to $\sum_{Y \supseteq X} f(Y)$. The second Möbius inversion formula gives the following inclusion-exclusion formula:

$$\left| \bigcap_{x \in X} A_x \cap \bigcap_{x \notin X} \overline{A}_x \right| = \sum_{Y \supseteq X} (-1)^{|Y|-|X|} \left| \bigcap_{y \in Y} A_y \right|.$$

Applying this equation to $X = \emptyset$ gives the usual inclusion-exclusion formula.

2. If $P = \mathbb{N}$, ordered by the usual $\leq$ relation, then

$$\mu(m, n) = \begin{cases} 1 & \text{if } m = n; \\ -1 & \text{if } m + 1 = n; \\ 0 & \text{in all other cases.} \end{cases}$$

The Möbius inversion formula implies

$$g(n) = \sum_{k \leq n} f(k) \quad \text{iff} \quad f(n) = g(n) - g(n-1).$$

3. If $P = \mathbb{N}$, ordered by the relation "divides", then the Möbius function is given by

$$\mu(m, n) = \overline{\mu}(n/m)$$

where $\overline{\mu}$ is the Möbius function known from number theory:

$$\overline{\mu}(n) = \begin{cases} 0 & \text{if some } m^2 > 1 \text{ divides } n, \\ (-1)^r & \text{if } n \text{ is square-free and the product of } r \text{ primes.} \end{cases}$$

The Möbius inversion formula implies

$$g(n) = \sum_{k|n} f(k) \quad \text{iff} \quad f(n) = \sum_{k|n} g(k)\overline{\mu}(n/k).$$

This is the Möbius inversion formula known in number theory.

4 Proof of the Möbius inversion formula

We show that $g(x) = S_{\leq} f(x)$ if and only if $f(x) = \sum_{y \leq x} g(y)\mu(y,x)$, the proof of the other statement is similar. The statement makes sense only if we assume that for any x only finitely many y's satisfy $y \leq x$. Let us fix an element x_0 and consider only elements that are less than or equal to x_0. We may assume this is our entire poset P. Let us associate to P a square matrix Z whose rows and columns are indexed with the elements of P, and which has a 1 in row x, column y exactly when $x \leq y$, and has a zero in all other rows. Multiplying the row vector $\underline{f} := (f(y) \,|\, y \in P)$ with the matrix Z from the right yields a row vector whose entry associated to x is $\sum_{y \leq x} f(y)$. Let us introduce $\underline{g} := (g(x) \mid x \in P)$, and the matrix M whose entry in row x and column y is $\mu(x,y)$ if $x \leq y$ and zero otherwise. The first statement in Theorem 2.1 is equivalent to saying

$$\underline{g} = \underline{f} * Z \quad \text{if and only if} \quad \underline{f} = \underline{g} * M.$$

This is obviously true if the matrices M and Z are inverses of each other, so it suffices to show

$$I = M * Z$$

To verify this we need to check that the product of a row indexed by x in M and a row indexed by y in Z is $\delta_{x,y}$, the Kronecker delta function. In other words we need to check

$$\sum_{x \leq z \leq y} \mu(x,z) = \delta_{x,y}$$

which is exactly the definition of the Möbius function.