

## The Fermat point

Let  $ABC$  be a triangle whose internal angles are all less than  $120^\circ$ . We define its Fermat point as the point  $P$  satisfying  $\angle APB = \angle BPC = \angle CPA = 120^\circ$ . Using this definition, the Fermat point clearly exist since it is the intersection of the (open) arc  $\{Q : \angle AQB = 120^\circ\}$  with the (open) arc  $\{Q : \angle BQC = 120^\circ\}$ .

Construct the regular triangles  $A'BC_\Delta$ ,  $AB'C_\Delta$ , and  $ABC'_\Delta$  as shown in Fig. 1.

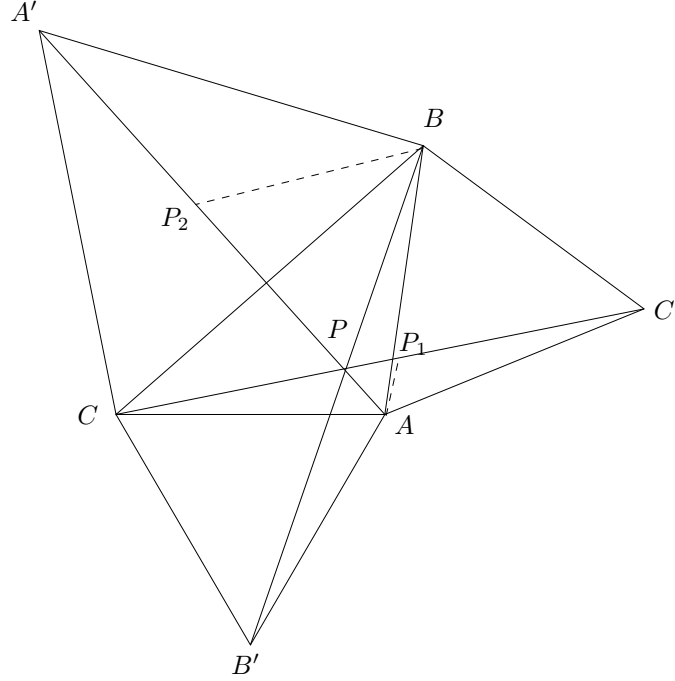


Figure 1: Constructing the Fermat point using regular triangles

**Proposition 1** *The lines  $AA'$ ,  $BB'$  and  $CC'$  meet in the Fermat point of the triangle.*

**Proof:** Rotation by  $60^\circ$  around  $A$  sends  $AB'B_\Delta$  into  $ACC'_\Delta$ . Introducing  $P$  for the intersection of  $B'B$  and  $C'C$  we see that  $\angle CPB' = 60^\circ$  and so  $\angle CPB = 120^\circ$ . Let us introduce  $P_1$  for the image of  $P$  under this rotation. Then  $|P_1C'| = |PB|$  by the congruence induced by the rotation, and  $|PP_1| = |PA|$  since  $PP_1A_\Delta$  is a regular triangle. Therefore

$$|CC'| = |CP| + |PP_1| + |P_1C'| = |PA| + |PB| + |PC|.$$

Consider now the rotation by  $60^\circ$  around  $B$ . This takes  $BC'C_\Delta$  into  $BAA'_\Delta$  and let us denote the image of  $P$  under this rotation by  $P_2$ . (Note that we know only that  $P$  is on  $CC'$ , we don't know yet whether it is also on  $AA'$ .) By the congruence  $BC'C_\Delta \cong BAA'_\Delta$  we have  $|AA'| = |CC'|$  and so  $|AA'| = |PA| + |PB| + |PC|$ . Observe also that  $|P_2A| = |PC|$  by the congruence induced by the rotation and that  $|PP_2| = |PB|$  since  $PP_2B_\Delta$  is a regular triangle. Therefore we get that

$$|A'P_2| + |P_2P| + |PA| = |PC| + |PB| + |PA| = |A'A|.$$

If  $P$  or  $P_2$  is not on the line  $AA'$  then (by the triangle inequality), the sum  $|A'P_2| + |P_2P| + |PA|$  is strictly greater than  $|A'A|$ , in contradiction with the above equality. Thus  $P$  is also on the line  $AA'$ . Now  $\angle CPA = 120^\circ$  follows in analogy with  $\angle CPB = 120^\circ$ .  $\diamond$

The first half of the proof of Proposition 1 is almost repeated in the proof of the following characterization of the Fermat point.

**Proposition 2** *The Fermat point minimizes the sum  $|QA| + |QB| + |QC|$  among all points  $Q$  in the plane.*

**Proof:** Consider any point  $Q$  in the plane that is different from the Fermat point. W.l.o.g. we may

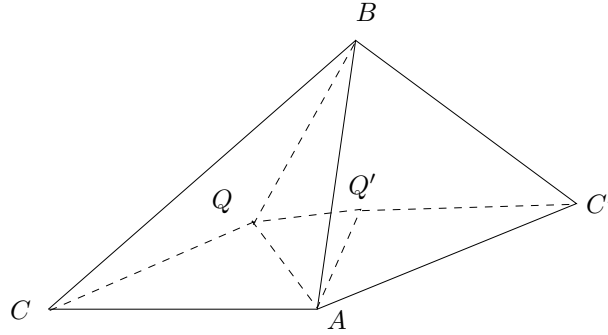


Figure 2: Extremal property of the Fermat point

assume that  $\angle AQC \neq 120^\circ$ . Let us rotate  $QAC_\Delta$  around  $A$  by  $60^\circ$ , as shown in Fig. 2. Denote the image of  $Q$  and  $B$  respectively by  $Q'$  and  $C'$  respectively. By rotational congruence we have  $|QB| = |Q'C'|$  and we also have  $|QQ'| = |AQ'|$  since  $AQQ'_\Delta$  is regular. Thus

$$|QA| + |QB| + |QC| = |CQ| + |QQ'| + |Q'C|.$$

Since  $\angle AQC \neq 120^\circ$ , the angle  $\angle AQQ' \neq 180^\circ$  and, by the triangle inequality,  $|CQ| + |QQ'| + |Q'C|$  is strictly more than  $|CC'|$ . In the proof of the previous proposition we have seen that the Fermat point satisfies  $|PA| + |PB| + |PC| = |CC'|$ .  $\diamond$