Assignment 12

Oral questions

1. Consider the fractional linear transformation $z \mapsto \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{R}$ and $ad-bc \neq 0$. Introduce $z = z_1+z_2 i$ and calculate explicitly the imaginary part of $\frac{az+b}{cz+d}$. Prove that the imaginary part of the image is positive for all $z_2 > 0$ if and only if $ad - bc > 0$.

   Now show that a conjugate fractional linear map $z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$ takes the upper half plane into itself if and only if $ad - bc < 0$.

2. Assume that the points A, B, C, D are either on the same line or on the same circle, and represent them with the complex numbers a, b, c, d. Prove that the cross ratio (AB, CD) equals $\frac{(a-c)(b-d)}{(c-b)(d-a)}$. (In particular, this expression of complex numbers is real!) *Hint:* Use the Star Trek Lemma.

Question to be answered in writing

1. Using that

$$\frac{az+b}{cz+d} = \begin{cases} \frac{a}{c} + \frac{b-ad/c}{cz+d} & \text{if } c \neq 0, \text{ and} \\ \frac{az+b}{d} & \text{if } c = 0, \end{cases}$$

   show that every fractional linear transformation arises as a combination of horizontal translations $z \mapsto z + b$, dilations $z \mapsto az$ and "reflected inversions" $z \mapsto 1/z$. Conclude that fractional linear transformations preserve angles and the cross-ratio.