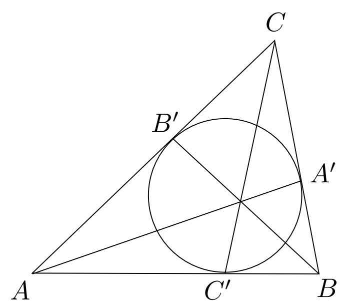
Assignment 6

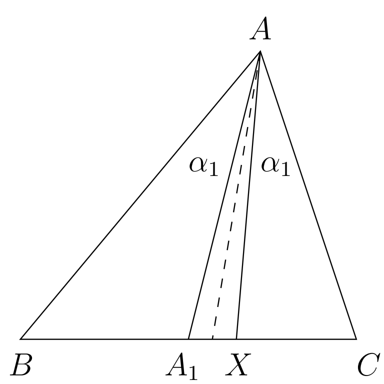
Oral questions

1. Prove that the Gergonne point exist by verifying that the conditions of Ceva's theorem are satisfied. (Hint: look for pairs of equal line segments.) Then explain how you could use Ceva's theorem to prove that the orthocenter exists. (These are two unrelated but very easy questions.)



2. Let X be the point where the symmedian from A intersects the line from BC. Prove that

$$\frac{BX}{XC} = \frac{AB^2}{AC^2}.$$

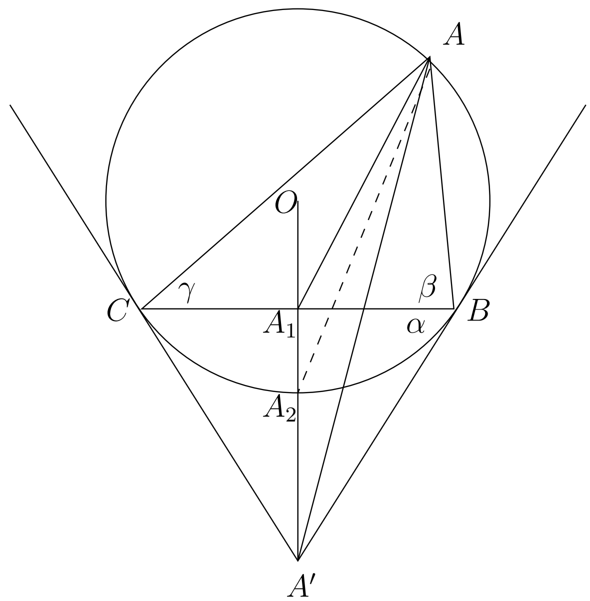


(Hint: introducing A_1 for the midpoint of BC, note that the angles marked α_1 are equal. Use the triangles from the proof of the second version of Ceva's theorem to translate proportions of line segments into proportions of areas, and use the similar relations for A_1 to get rid of $\sin(\alpha_1)$ and $\sin(\alpha - \alpha_1)$. Here α is the angle $\angle BAC$.)

3. Use Ceva's theorem to prove that the Nagel point exists. (See your notes for the definition.)

Question to be answered in writing

1. As the key to prove that the Lemoine point of the Gergonne triangle is the Gergonne point of the original triangle, verify the following statement.



Consider a triangle $ABC_\triangle$ with circumcenter O, circumradius r, angles α, β and γ. Draw tangents to the circumcircle at B and C and let A' be the intersection of these tangents. Let A_1 be the midpoint of the line segment BC and A_2 be the intersection of the angle bisector of $\angle BAC$ with the circumcircle. Prove that AA_2 is also the angle bisector of $\angle A_1AA'$.

Hints: Express all distances as functions of the sines and cosines of the angles and r. Calculate $|OA_1|$ (from $OA_1B_\triangle$), $|A_1A_2| = |OA_2| - |OA_1|$, $|A_1A'|$ (from $A'A_1B_\triangle$), and $|A_2A'| = |A_1A'| - |A_1A_2|$. Calculate the proportion of $|A_2A'|$ and $|A_1A_2|$. This is the end of the first phase. In the second phase, calculate $|A'B|$ (from $A'A_1B_\triangle$), $|AA'|^2$ (from $A'AB_\triangle$) and $|A_1A|^2$ (from $A_1AB_\triangle$). Calculate the proportion of $|AA'|^2$ and $|A_1A|^2$ and **now** take the square root. If you did everything correctly then you will find

$$\frac{|A_2A'|}{|A_1A_2|} = \frac{|AA'|}{|A_1A|} = 1/\cos(\alpha)$$

and so you will be done by applying the angle bisector theorem to $A_1AA'_\triangle$.