Assignment 5

Oral questions

1. Prove the following converse of the Star Trek Lemma: given a circle centered at O, and three points A, B, and C such that

   (i) B and C are on the circle,
   (ii) The angle $\angle BAC$ is the half of $\angle BOC$,

   the angle A is also on the circle. (I ask you to work out only the case when $\angle BAC$ is acute and O lies in its interior, keeping in mind that there are also other cases, see the first written question. It might help if you consider, how $\angle BAC$ changes when you move the point A on a line containing O, towards O or away from it.)

2. Prove that a quadrilateral is cyclic if and only if the sum of two of its opposite angles is 180^o. Explain which implication is related to the Star Trek Lemma, and which to its converse.

3. Let a, b, and c be the sides of a triangle, and A its area. Prove that the excircle at side a has radius $2A/(-a+b+c)$.

Questions to be answered in writing

1. Draw a picture for each possible instance of the Star Trek Lemma. Distinguish cases depending upon whether the central angle is convex, concave, or 180^o, and whether the center of the circle is in the interior, on the boundary or outside the inscribed angle. Without completing the proof for each case, indicate on each picture the isoceles triangles you would use to prove the lemma.

2. Assume that each angle of the triangle $ABC_\triangle$ is less than 120^o. Consider any point P inside $ABC_\triangle$. Prove that $PA + PB + PC$ is minimal when P is the Fermat point. *Hint:* Rotate around A by 60^o. C goes into C', P goes into P'. Using these new points observe that $PA + PB + PC$ is equal to the length of a walk, consisting of three line segments, from B to C'. The shortest walk is the straight line.