Name: **Student ID:**

Sample Test I.

The actual test will have 5 questions and perhaps one bonus question. You will have 50 minutes to answer them, without using your notes or communicating with other students. You will have to give the simplest possible answer and show all your work.

1. Prove by induction that
$$1 + 3 + 5 + \ldots + 2n - 1 = n^2$$
holds for every positive integer n.

2. Evaluate the following sums using the binomial theorem.
   (a) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} (n-1)^k$.
   (b) $\displaystyle\sum_{k=0}^{n} \binom{2n}{2k} a^{2k} \cdot b^{2n-2k}$. (Hint: Try to write this sum as the average of two sums.)

3. Prove that the relation "divides" is a partial order on positive integers.

4. Prove the correct statement, provide a counterexample to the false one:
   (a) If $d|a$ and $d|b$ then $d|a+b$.
   (b) If $d|a+b$ then $d|a$ and $d|b$.

5. In which sense is the greatest common divisor of two integers the "greatest"?

6. Using Euclid's Algorithm, find the greatest common divisor of 540 and 246 and write it in the form $540x + 246y$. (Note: You will not get full credit unless you record the individual steps of the algorithm.)

7. What is the relation between the greatest common divisor $\gcd(a, b)$ and the least common multiple $\operatorname{lcm}(a, b)$ of the integers a and b? Prove your claim.

8. Solve the Diophantine equation $6x - 21y = 15$.

9. State and prove Euclid's lemma.

10. Which principle of induction do you have to use to prove the existence part of the fundamental theorem of arithmetic? Explain why.

11. Explain how to use Euclid's lemma in the proof of the fundamental theorem of arithmetic.

12. Prove that there are infinitely many primes.

13. Prove that congruence is an equivalence relation.

14. Prove that congruence is compatible with addition, subtraction, and multiplication.

15. Is congruence compatible with division? In which sense?

16. Find the number between 0 and 6 that is congruent to 3^{2335} modulo 7.

17. Prove that a number is divisible by 9 if and only if the sum of its digits is. Use this result to decide whether 13273245409265 is divisible by 9.

18. Prove that the decimal representation of positive integers is unique.

19. Given an integer $b > 1$, prove that every number has a base b representation. Illustrate your proof by finding the base 3 representation of 251.

20. Solve the following congruences:

    (a) $3x \equiv 2 \pmod{7}$

    (b) $2x \equiv 8 \pmod{10}$

    (c) $14x \equiv 3 \pmod{21}$

Good luck. Gábor Hetyei