Sample Test 2

The real test will have less questions and you will have about 75 minutes to answer them. The usage of books or notes, or communicating with other students will not be allowed. You will have to give the simplest possible answer and show all your work. The questions below are sample questions related to stating and proving theorems. Besides trying to answer these questions, make sure you also review all homework exercises. The test may also have questions similar to those exercises. During the test, the usage of books or notes, or communicating with other students will not be allowed.

1. Assume p is a prime and a is not a multiple of p. Prove that the congruence $ax \equiv 1 \pmod{p}$ has a solution.
2. Which of the following is a subring of $\mathbb{Z}$: the set of even integers, or the set of odd integers? Justify your answer!
3. Is the Cartesian product of two integral domains an integral domain? Justify your answer!
4. What is $a \cdot 0_R$ equal to in a ring? Prove your claim!
5. Prove that $-(-a) = a$ in a ring.
6. Prove that $-(a+b) = (-a) + (-b)$ in a ring.
7. Prove that a subset S is a subring if it is not empty, and it is closed under subtraction and multiplication.
8. If $ac = bc$ in a ring, does it always follow that $a = b$? When does it follow? Justify your claim with example and/or proof, as appropriate.
9. Prove that every field is an integral domain. When is the converse true? Justify your answer!
10. Give an example of a zero divisor and of an idempotent element.
11. Let $f : R \to S$ be a ring homomorphism. Prove that $f(0_R) = 0_S$ and that $f(-a) = -f(a)$ for all $a \in R$.
12. Is the map $f : \mathbb{Q} \to \mathbb{Q}$ sending x into $\frac{1}{1+x^2}$ a homomorphism? Justify your answer!
13. Prove that conjugation $\overline{} : \mathbb{C} \to \mathbb{C}$, sending $a + bi$ into $\overline{a+bi} = a - bi$, is an *automorphism* of $\mathbb{C}$.
14. Let R be a ring. When is it true that $\deg(f \cdot g) = \deg(f) + \deg(g)$ holds for all nonzero polynomials $f, g \in R[x]$?
15. Let F be a field. Describe the units of $F[x]$. Justify your description.
16. State the division algorithm theorem in $F[x]$ and prove the uniqueness part.

17. State the division algorithm theorem in $F[x]$ and outline the proof of the existence part.
18. Define the greatest common divisor of two polynomials in $F[x]$ and explain how the Euclidean algorithm may be used to find it. (You do not have to prove your claim.)
19. Explain why every common divisor of two polynomials in $F[x]$ divides their greatest common divisor.
20. Explain why the greatest common divisors of two polynomials $a(x)$ and $b(x)$ in $F[x]$ may be written as $u(x) \cdot a(x) + v(x) \cdot b(x)$ for some polynomials $u(x)$ and $v(x)$.
21. Prove the following variant of Euclid's lemma for polynomials: given an irreducible polynomial $p(x) \in F[x]$, if $p(x)$ divides $a(x) \cdot b(x)$ then $p(x)$ divides $a(x)$ or $b(x)$.
22. Let F be a field. Prove that every nonconstant polynomial in $F[x]$ is the product of finitely many irreducible polynomials.

Good luck. Gábor Hetyei