Sample Final Exam Questions.

This study guide is subject to updates until our lass class on Wednesday December 8, 2021

Last update: December 3, 2021

The actual final exam will have a mandatory and an optional section. The optional questions will be similar to the ones on the previous (sample) tests, and need to be answered only if you do not want me to re-use your average test score. The questions below are supposed to help you prepare for the mandatory part of the final. Besides trying to answer these questions, make sure you also review all homework exercises.

In all sample questions F denotes a field.

1. Let F be a field. Prove that every nonconstant polynomial in $F[x]$ is the product of finitely many irreducible polynomials.
2. State and prove the remainder theorem for polynomials in $F[x]$.
3. State and prove the factor theorem for polynomials with $F[x]$.
4. Explain why reducible polynomials of degree at most 3 in $F[x]$ must have a root.
5. State the fundamental theorem of algebra.
6. Prove that the map $a + bi \mapsto a - bi$, sending each complex number into its conjugate is an automorphism of the ring of complex numbers $\mathbb{C}$.
7. Use the previous two statements to show that irreducible polynomials in $\mathbb{R}(x)$ have degree at most two.
8. State and prove the rational zeros (rational root test) theorem.
9. Find all rational zeros of the polynomial $10x^4 + 7x^3 + 6x^2 - 4x - 1$.
10. Let F be a field and $p(x)$ be a nonzero polynomial in $F[x]$. Define congruence modulo $p(x)$ and prove it is an equivalence relation.
11. Given $p(x) \in F[x]$ as in the previous question, prove that congruence modulo $p(x)$ is compatible with the ring operations.
12. Assume $p(x) \in F[x]$ has degree n, where n is a positive integer. Prove that every congruence class modulo $p(x)$ may be represented by a polynomial of degree less than n, and show that this representative is unique. (We consider the constant 0 polynomial as a polynomial of degree $-\infty$.)
13. Assume $p(x) \in F[x]$ has positive degree. Prove that the set $F[x]/(p(x))$ of congruence classes is a commutative ring with identity that contains a subring that is isomorphic to F. (You may use the previous statement in your proof.)

14. Assume $p(x) \in F[x]$ has positive degree and that $f(x) \in F[x]$ is relative prime to $p(x)$. Prove that the class of $f(x)$ is a unit in $F[x]/(p(x))$.

15. Assume $p(x) \in F[x]$ is an irreducible polynomial. Explain how the previous statement implies that $F[x]/(p(x))$ is a field.

16. Assume $p(x) \in F[x]$ is a polynomial of positive degree and that $F[x]/(p(x))$ is an integral domain. Prove that $p(x)$ is irreducible.

17. Assume $p(x) \in F[x]$ is an irreducible polynomial. Prove that the extension field $F[x]/(p(x))$ contains a root of $p(x)$.

18. Define an ideal, and prove that even integers are an ideal of $\mathbb{Z}$.

19. Assume R is a commutative ring with a multiplicative identity. Describe the ideal generated by a finite subset $\{c_1, \ldots, c_n\}$ of R, prove that it is an ideal, containing the set $\{c_1, \ldots, c_n\}$.

20. Define congruence modulo an ideal and prove that it is an equivalence relation.

21. Prove that congruence modulo an ideal is compatible with the ring operations.

22. Let $f : R \to S$ be a ring homomorphism. Define the kernel of f, and prove that it is an ideal.

23. Let $f : R \to S$ be a ring homomorphism and $a, b \in R$. Describe, in terms of the kernel of f, when does $f(a)$ equal $f(b)$.

24. Let R be a ring and I an ideal of this ring. Describe the natural homomorphism from R to the ring R/I. (No need to prove that this is a homomorphism.)

25. Let $a \in F$ be a fixed field element. Describe the kernel of the evaluation homomorphism $\phi_a : F[x] \to F$, sending each $f(x) \in F[x]$ into $f(a)$.

26. State the first isomorphism theorem.

Good luck. Gábor Hetyei