Sample Test II.

The real test will have less questions and you will have about 75 minutes to answer them. The usage of books or notes, or communicating with other students will not be allowed. You will have to give the simplest possible answer and show all your work. The questions below are sample questions related to stating and proving theorems. Besides trying to answer these questions, make sure you also review all homework exercises. The test may also have questions similar to those exercises. During the test, the usage of books or notes, or communicating with other students will not be allowed.

1. Which of the following is a subring of $\mathbb{Z}$: the set of even integers, or the set of odd integers? Justify your answer!
2. Is the Cartesian product of two integral domains an integral domain? Justify your answer!
3. Let R be a ring and let a be any ring element. Prove that the solution x of the equation $a + x = 0_R$ is unique. Explain how this may be used to prove that $a + b = a + c$ implies $b = c$.
4. What is $a \cdot 0_R$ equal to in a ring? Prove your claim!
5. Prove that $-(-a) = a$ in a ring.
6. Prove that $-(a + b) = (-a) + (-b)$ in a ring.
7. Prove that a subset S is a subring if it is not empty, and it is closed under subtraction and multiplication.
8. Describe the unique solution of the equation $a + x = b$ in a ring.
9. If $ac = bc$ in a ring, does it always follow that $a = b$? When does it follow? Justify your claim with example and/or proof, as appropriate.
10. Prove that every field is an integral domain. Is the converse true?
11. Give an example of a zero divisor and an idempotent element.
12. Let $f : R \to S$ be a ring homomorphism. Prove that $f(0_R) = 0_S$ and that $f(-a) = -f(a)$ for all $a \in R$.
13. Is the map $f : \mathbb{Q} \to \mathbb{Q}$ sending x into $\frac{1}{1+x^2}$ a homomorphism? Justify your answer!
14. Prove that the map $\mathbb{Z} \to \mathbb{Z}_5$ sending each $n \in \mathbb{Z}$ into its congruence class $[n]$ is a surjective homomorphism.
15. (Potential bonus question) Prove that $\mathbb{Z}_6$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$.
16. (Potential bonus question) Prove that $\mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

17. Let R be a ring. When is it true that $\deg(f \cdot g) = \deg(f) + \deg(g)$ holds for all nonzero polynomials $f, g \in R[x]$?

18. Let F be a field. Describe the units of $F[x]$. Justify your description.

19. State the division algorithm theorem in $F[x]$ and prove the uniqueness part.

20. State the division algorithm theorem in $F[x]$ and prove the existence part.

21. Define the greatest common divisor of two polynomials in $F[x]$ and explain how the Euclidean algorithm may be used to find it. (You do not have to prove your claim.)

22. Explain why every common divisor of two polynomials in $F[x]$ divides their greatest common divisor.

23. Explain why the greatest common divisors of two polynomials $a(x)$ and $b(x)$ in $F[x]$ may be written as $u(x) \cdot a(x) + v(x) \cdot b(x)$ for some polynomials $u(x)$ and $b(x)$.

Good luck. Gábor Hetyei