Sample Test I.

Version of Friday, September 14, 2012

This document is subject to minor updates till Tuesday, September 18, 2012.

The real test will have less questions and you will have about 75 minutes to answer them. The usage of books or notes, or communicating with other students will not be allowed. You will have to give the simplest possible answer and show all your work. The questions below are sample questions related to stating and proving theorems. Besides trying to answer these questions, make sure you also review all homework exercises. The test may also have questions similar to those exercises.

1. Give an example of a totally ordered set that is not well-ordered.
2. Define the quotient and the remainder arising when the positive integer a by the positive integer b. (Write down the defining equation, and state the properties of the quotient and the remainder.)
3. Prove that for every positive integer a and every positive integer b there is a quotient q and a remainder r satisfying $a = qb + r$ and $0 \leq r < b$.
4. Prove the uniqueness of the quotient and the remainder.
5. Prove that the relation "divides" is a partial order on the set of positive integers.
6. Is the relation "divides" an equivalence relation on integers? Justify your answer!
7. Define the greatest common divisor of two integers a and b and prove that every other common divisor divides the greatest common divisor.
8. Describe Euclid's algorithm and explain how it may be used to find the greatest common divisor.
9. Prove that the greatest common divisor of two integers a and b may be written as an integer linear combination of a and b.
10. Define primes and show that they satisfy the property stated in Euclid's lemma.
11. Explain how Euclid's lemma may be used to prove the uniqueness of prime factorization.
12. Prove that every integer may be written as a product of primes.
13. Prove that $\sqrt{2}$ is irrational.
14. Let $n > 1$ be a positive integer. Define congruence modulo n and prove it is an equivalence relation.
15. Prove that congruence is compatible with addition and multiplication.

16. Explain how the compatibility of congruence with addition and multiplication allows us to define these operations on congruence classes.
17. Explain why there are exactly n congruence classes modulo n.
18. Assume p is a prime and a is not a multiple of p. Prove that the congruence $ax \equiv b \pmod{p}$ has a solution for any integer b.
19. Assume p is a prime and a is not a multiple of p. Prove that the the solution of the congruence $ax \equiv b \pmod{p}$ is unique.
20. Generalize the previous statement as follows: replace the prime p with an arbitrary integer n and state a necessary and sufficient condition the the congruence $ax \equiv b \pmod{n}$ to have a solution. Prove your generalization.
21. Give an example of a ring that is not an integral domain.

Good luck. Gábor Hetyei