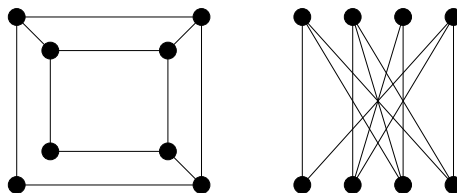# Sample Test I.

The real test will have less questions and you will have about 80 minutes to answer them. The usage of books or notes, or communicating with other students will not be allowed. You will have to give the simplest possible answer and show all your work.

1. Are the two graphs shown in the picture isomorphic? Justify your claim.

2. State and prove the degree-sum formula.

3. Explain how the degree-sum formula may be used to prove that a "mountain climber puzzle" always has a solution.

4. What is the number of edges in the n-dimensional hypercube?

5. What is the number of edges in the complete bipartite graph $K_{m,n}$?

6. Give a necessary and sufficient condition for a graph to be bipartite and outline the proof of your claim.

7. A planar graph has 6 vertices and 3 regions. What is the number of its edges?

8. Prove that in a planar graph the number e of edges does not exceed $3v - 6$, where v is the number of vertices.

9. Use the result in the previous question to show that K_5 is not planar.

10. Use the circle-chord method to show that $K_{3,3}$ is not planar.

11. State the 4-color theorem and explain how it applies to map coloring. In particular, does it mean that every map can be colored using four colors?

12. Give a necessary and sufficient condition for the existence of an Eulerian cycle in a graph. Outline the proof.

13. For which values of n does the complete graph K_n have an Eulerian cycle?

14. The maximum degree in a graph is d. What can you say about its edge chromatic number?

15. Grinberg's theorem states that a planar graph that has a Hamilton circuit, satisfies the formula:

$$\sum_i (i-2)(r_i - r_i') = 0.$$

Here r_i resp. r_i' is the number of regions that have i sides and lay inside resp. outside the Hamilton circuit.

Use Grinberg's theorem to show that the graph shown below has no Hamiltonian circuit.
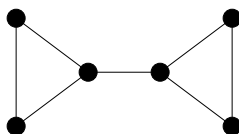
16. Now use the three simple rules about building a Hamilton circuit to show that the graph in the previous question has no Hamilton circuit.

17. What is the chromatic number of a wheel graph with n vertices?

18. The maximum degree of a vertex in a graph G is 5. What can you say about the minimum number of colors needed to properly color its edges?

19. According to Brooks theorem which are the only graphs whose chromatic number is larger than their maximum degree?

20. What is the chromatic polynomial of a path of length n? Prove your formula.

21. Prove that the chromatic polynomial $P_k(G)$ of a graph G satisfies the recurrence

$$P_k(G) = P_k(G - e) - P_k(G/e).$$

Here e is an edge of G, and $G - e$, respectively G/e are the graphs obtained by deleting, respectively, contracting the edge e. Explain, how the above recurrence may be used to prove that $P_k(G)$ is a polynomial. (What is the basis for the induction?)

22. Outline the proof of the fact that every planar graph may be properly colored using at most 5 colors.

23. Explain why each tree must have at least one leaf. (Also define leafs.)

24. State the number of edges in a tree on n vertices and prove your formula using induction and the statement in the previous question.

Good luck.   Gábor Hetyei