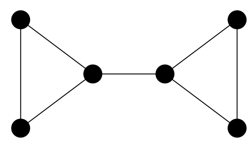
Answers to select questions on Sample Test I.

13. Grinbergs' theorem states that a planar graph that has a Hamilton circuit, satisfies the formula:

$$\sum_i (i-2)(r_i - r_i') = 0.$$

Here r_i resp. r_i' is the number of regions that have i sides and lay inside resp. outside the Hamilton circuit.

Use Grinbergs' theorem to show that the graph shown below has no Hamiltonian circuit.



Answer: The graph in the picture has one region with 7 sides and two regions with 3 sides. Thus it must satisfy

$$r_3 + r_3' = 2 \quad \text{and} \quad r_7 + r_7' = 1.$$

Without loss of generality we may assume $r_7 = 1$ and $r_7' = 0$. Grinbergs' theorem requires then

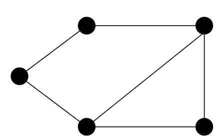
$$(r_3 - r_3') + 5(1-0) = 0.$$

Thus r_3 and r_3' is the solution of the system of equations

$$\begin{array}{rcl} r_3 + r_3' & = & 2 \\ r_3 - r_3' & = & -5 \end{array}$$

This gives $r_3 = -3/2$ which is not possible since r_3 and r_3' must be non-negative integers.

17. Find the chromatic polynomial of the graph shown in the picture.

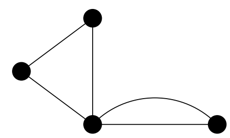


Answer: $k(k-1)^4 - k(k-1)^3 + k(k-1)(k-2) - k(k-1)^2(k-2)$.

We use the formula

$$P_k(G) = P_k(G \setminus e) - P_k(G/e) \tag{1}$$

repeatedly. (Corollary of Theorem 6 on page 82). The graph shown in the picture is planar, its infinite outside region is bounded by a 5-cycle, and there is a chord inside this 5-cycle. Let us apply formula (1) to this chord. Removing the chord gives a 5-cycle, contracting the chord gives the following graph:



The chromatic polynomial of this second graph is $k(k-1)^2(k-2)$. In fact, the only vertex of degree 4 may be colored k, ways, one of its neighbors on the 3-cycle, $(k-1)$ ways, the other $(k-2)$ ways, the remaining vertex may be colored $(k-1)$ ways. We are left to calculate the chromatic polynomial of a 5-cycle C_5. Applying formula (1) to any edge of C_5 gives

$$P_k(C_5) = P_k(P_4) - P_k(C_4)$$

where P_5 is the path of length 5 and C_4 is a 4-cycle. We know that $P_k(P_4) = k(k-1)^4$, and for $P_k(C_4)$ we get

$$P_k(C_4) = P_k(P_3) - P_k(C_3).$$

Here $P_k(P_3) = k(k-1)^3$ and $P_k(C_3) = k(k-1)(k-2)$. Working our way back we get

$$P_k(C_4) = k(k-1)^3 - k(k-1)(k-2),$$

(see also Example 1(c) on page 83), and

$$P_k(C_5) = k(k-1)^4 - k(k-1)^3 + k(k-1)(k-2).$$

Finally, the chromatic polynomial of the graph in the question is

$$P_k(C_5) - k(k-1)^2(k-2) = k(k-1)^4 - k(k-1)^3 + k(k-1)(k-2) - k(k-1)^2(k-2).$$

18. What is the chromatic polynomial of a path of length n? Prove your formula.

    Answer: $k(k-1)^n$

    Let us color the vertices along the path, one by one, starting from one end. There are k ways to color the first vertex. For each subsequent vertex the only restriction is that its color can not be identical to the color of its already colored neighbor. Thus we have $(k-1)$ ways to color each subsequent vertex.

Last update: Friday, September 21, 2007