THE MEASURE ALGEBRA DOES NOT ALWAYS EMBED

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Abstract. The Open Colouring Axiom implies that the measure algebra cannot be embedded into $\mathcal{P}(\mathbb{N})/\mathit{fin}$. We also discuss the errors in previous results on the embeddability of the measure algebra.

Introduction

The aim of this paper is to prove the following result.

Main Theorem. *The Open Colouring Axiom implies that the measure algebra cannot be embedded into the Boolean algebra $\mathcal{P}(\mathbb{N})/\mathit{fin}$.*

By 'the measure algebra' we mean the quotient of the σ-algebra of Borel sets of the real line by the ideal of sets of measure zero.

There are various reasons, besides sheer curiosity, why it is of interest to know whether the measure algebra can be embedded into $\mathcal{P}(\mathbb{N})/\mathit{fin}$. One reason is that there is great interest in determining what the subalgebras of $\mathcal{P}(\mathbb{N})/\mathit{fin}$ are. One of the earliest and most influential result in this direction is Parovičenko's theorem from [11], which states that every Boolean algebra of size $\aleph_1$ can be embedded into $\mathcal{P}(\mathbb{N})/\mathit{fin}$ — with the obvious corollary that the Continuum Hypothesis (CH) implies that $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is a universal Boolean algebra of size $\mathfrak{c}$: a Boolean algebra embeds into $\mathcal{P}(\mathbb{N})/\mathit{fin}$ iff it is of size $\mathfrak{c}$ or less. It is therefore natural to ask how much of this universality remains without assumptions beyond ZFC. It has long been known that every σ-centered Boolean algebra embeds into $\mathcal{P}(\mathbb{N})/\mathit{fin}$ but the question for more general c.c.c. Boolean algebras has proven to be much more difficult — with the case of the measure algebra being seen as a touchstone.

This particular case was especially interesting since, by Stone duality, an embedding of the measure into $\mathcal{P}(\mathbb{N})/\mathit{fin}$ would provide a c.c.c. nonseparable remainder of $\mathbb{N}$. A ZFC construction of such a remainder was given by Bell in [2]; such remainders were put to good use by Van Mill in [17], see also his survey [18]. The question of the embeddability of the measure algebra remained however.

In recent years many results about maps between Boolean algebras or topological spaces, which were shown to hold under CH, were shown to fail under OCA — see for example [8, 5, 4]. Our result and its proof fall into the same category: in all cases OCA implies that the desired map should have some simple structure whereas one can show, usually in ZFC, that the desired map cannot have this simple structure.

We should note that our result clashes with the result of Frankiewicz and Gutek from [7], which says that Martin's Axiom implies that the measure algebra can

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be embedded into $\mathcal{P}(\mathbb{N})/\mathit{fin}$; for completeness we shall point out a gap in their proof. In addition we feel that we should mention the paper [6]. The main result in that paper is that one can establish the consistency of the nonembeddability of the measure algebra using Shelah's oracle-c.c. method. Regrettably, again the argumentation appears to be incomplete as we shall discuss further in Section 5.

The paper is organized as follows. Section 1 contains the necessary preliminaries, including a discussion of what 'simple structure' means in our context. Section 2 shows that no embedding can have this simple structure. In Sections 3 and 4 we show how OCA implies that embeddings must have a simple structure. Finally then Section 5 is devoted to the discussion of problems in the previously published work in this area. This will allow us to resurrect the interesting question of whether this result can be established with the oracle-c.c. method.

1. Preliminaries

The Measure Algebra. The standard representation of the Measure Algebra is as the quotient of the σ-algebra of Borel sets of the unit interval by the ideal of sets of Lebesgue measure zero. For ease of notation we choose a different underlying set, namely $\mathbb{C} = \omega \times 2^\omega$, where 2^ω is the Cantor set. We consider the Cantor set endowed with the natural coin-tossing measure μ, determined by specifying $\mu\bigl([s]\bigr) = 2^{-|s|}$. Here s denotes a finite partial function from ω to 2 and $[s] = \{x \in 2^\omega : s \subset x\}$. We extend μ on the Borel sets of $\mathbb{C}$ by setting $\mu\bigl(\{n\} \times [s]\bigr) = 2^{-|s|}$ for all n and s.

The measure algebra is isomorphic to the quotient algebra $\mathbb{M} = \mathrm{Bor}(\mathbb{C})/\mathcal{N}$, where $\mathcal{N} = \{N \subseteq \mathbb{C} : \mu(N) = 0\}$; henceforth we shall work with $\mathbb{M}$.

Liftings of embeddings. Assume $\varphi : \mathbb{M} \to \mathcal{P}(\mathbb{N})/\mathit{fin}$ is an embedding of Boolean algebras and take a *lifting* $\Phi : \mathbb{M} \to \mathcal{P}(\mathbb{N})$ of φ; this is a map that chooses a representative $\Phi(a)$ of $\varphi(a)$ for every a in $\mathbb{M}$.

We shall be working mostly with the restrictions of φ and Φ to the family of (equivalence classes of) open subsets of $\mathbb{C}$ and in particular with their restrictions to the canonical base for $\mathbb{C}$, which is

$$\mathcal{B} = \bigl\{\{n\} \times [s] : n \in \omega, s \in 2^{<\omega}\bigr\}.$$

To keep our formulas manageable we shall identify $\mathcal{B}$ with the set $B = \omega \times 2^{<\omega}$. We shall also be using layers/strata of B along functions from ω to ω: for $f \in {}^\omega\omega$ we put $B_f = \bigl\{\langle n, s\rangle : n \in \omega, s \in 2^{f(n)}\bigr\}$.

For a subset O of B_f we abbreviate $\varphi\bigl(\bigcup\bigl\{\{n\} \times [s] : \langle n, s\rangle \in O\bigr\}\bigr)$ by $\varphi(O)$ and define $\Phi(O)$ similarly. Observe that $O \mapsto \varphi(O)$ defines an embedding of $\mathcal{P}(B_f)$ into $\mathcal{P}(\mathbb{N})/\mathit{fin}$. As an extra piece of notation we use $\Phi[O]$ (square brackets) to denote the union $\bigcup\bigl\{\Phi(n, s) : \langle n, s\rangle \in O\bigr\}$, where $\Phi(n, s)$ abbreviates $\Phi\bigl(\bigl\{\langle n, s\rangle\bigr\}\bigr)$.

For later use we explicitly record the following easy lemma.

Lemma 1.1. *If $f \in {}^\omega\omega$ and if O is a finite subset of B_f then $\Phi(O) =^* \Phi[O]$.*

Proof. Both sets represent $\varphi(O)$. $\square$

Let us call a lifting *complete* if it satisfies Lemma 1.1 for *every* $f \in {}^\omega\omega$ and *every* subset O of B_f.

We can always make a lifting Φ *exact*, by which we mean that the sets $\Phi(n, \emptyset)$ form a partition of $\mathbb{N}$ and that every $\Phi(s, n)$ is the disjoint union of $\Phi(n, s^\frown 0)$

and $\Phi(n, s^\frown 1)$; indeed, we need only change each of the countably many sets $\Phi(n, s)$ by adding or deleting finitely many points to achieve this.

Our proof may now be summarized in a few lines:

(1) For every exact lifting Φ of an embedding φ there are an $f \in {}^\omega\omega$ and an infinite subset O of B_f such that $\Phi(O) \neq^* \Phi[O]$, i.e., no exact lifting is complete — see Proposition 2.1.

(2) OCA implies that every embedding φ gives rise to an embedding ϕ with a lifting Φ that is both exact and complete — see Section 3.

Together these two statements show that under OCA there cannot be any embedding of $\mathbb{M}$ into $\mathcal{P}(\mathbb{N})/\mathit{fin}$ at all.

The Open Colouring Axiom. The *Open Coloring Axiom* (OCA) was formulated by Todorčević in [15]. It reads as follows: if X is separable and metrizable and if $[X]^2 = K_0 \cup K_1$, where K_0 is open in the product topology of $[X]^2$, then *either* X has an uncountable K_0-homogeneous subset Y *or* X is the union of countably many K_1-homogeneous subsets.

One can deduce OCA from the *Proper Forcing Axiom* (PFA) or prove it consistent in an ω_2-length countable support proper iterated forcing construction, using $\diamondsuit$ on ω_2 to predict all possible subsets of the Hilbert cube and all possible open colourings of these.

We shall not apply OCA directly but use some of its known consequences to prove our main result. A major application occurs in Section 4, where we rely on a result from [4] regarding the behaviour of embeddings of $\mathcal{P}(\mathbb{N})/\mathit{fin}$ into itself.

Here and in the next subsection we collect some results of a more general nature. To begin a definition: $\mathfrak{b}$ is the minimum cardinality of a family $\mathcal{F}$ of functions from $\mathbb{N}$ to $\mathbb{N}$ for which there is no upper bound with respect to $\leqslant^*$, i.e., whenever $g \in \mathbb{N}^\mathbb{N}$ there if $f \in F$ such that $\bigl\{n : f(n) > g(n)\bigr\}$ is infinite.

The first consequence of OCA that we need is the equality $\mathfrak{b} = \aleph_2$; it was established in [1, Theorem 3.16]. It follows that the following lemma also holds under OCA.

Lemma 1.2 ($\mathfrak{b} \geqslant \aleph_2$). *Assume $f \mapsto \alpha_f$ is a map from ${}^\omega\omega$ to ω_1 that is monotone with respect to $\leqslant^*$ and $\in$, i.e., if $f \leqslant^* g$ then $\alpha_f \leqslant \alpha_g$. Then the map is bounded, i.e., there is an α such that $\alpha_f \leqslant \alpha$ for all f.*

Proof. Because $\mathfrak{b} \geqslant \aleph_2$ there is a $\leqslant^*$-cofinal family $\mathcal{F}$ in ${}^\omega\omega$ on which our map constant, say with value α. Because the map is monotone this α is the ordinal that we are looking for. $\square$

Coherent families of functions. Twice in our proof we shall want to combine a family of partial functions into one single function. In both cases we shall have an ideal $\mathcal{I}$ of subsets of some countable set C and for each I a function f_I with domain I such that whenever $I \subseteq J$ in $\mathcal{I}$ we have $f_J \restriction I =^* f_I$ — such a family will be called *coherent*. The following theorem, which is Theorem 3.13 from [1], tells us when a coherent family can be *uniformized*, i.e., when we can get one function f with domain $\bigcup \mathcal{I}$ such that $f \restriction I =^* f_I$ for all I.

Theorem 1.3 (OCA). *If $\mathcal{I}$ is a $P_{\aleph_1}$-ideal then every coherent family of functions on $\mathcal{I}$ with values in ω can be uniformized.*

An ideal $\mathcal{I}$ is a $P_{\aleph_1}$-ideal if for every subfamily $\mathcal{I}'$ of $\mathcal{I}$ of size $\aleph_1$ (or less) one can find an element J of $\mathcal{I}$ such that $I \subseteq^* J$ for all $I \in \mathcal{I}'$.

We shall need the following generalization of Theorem 1.3 — it actually turns out to be a special case.

Theorem 1.4 (OCA)**.** *If $\mathcal{I}$ is a $P_{\aleph_1}$-ideal on ω then every coherent family of functions on $\mathcal{I}$ with values in $\mathcal{P}(\mathbb{N})$ can be uniformized.*

Proof. Let $\{f_I : I \in \mathcal{I}\}$ be a coherent family of functions, with values in $\mathcal{P}(\mathbb{N})$. For $I \in \mathcal{I}$ and $g \in {}^\omega\omega$ let $L_{I,g} = \bigl\{\langle n, m\rangle : n \in I, m \leqslant g(n)\bigr\}$ and $R_{I,f} = \bigl\{\langle n, m\rangle : n \in I, m \leqslant g(n), m \in f_I(n)\bigr\}$. The sets $L_{I,g}$ generate a $P_{\aleph_1}$-ideal on the countable set $\mathbb{N}\times\mathbb{N}$ and one readily checks that $R_{I,g} =^* R_{J,h} \cap L_{I,g}$ whenever $I \subseteq J$ and $g <^* h$.

Now apply Theorem 1.3 to find $R \subseteq \mathbb{N}\times\mathbb{N}$ such that $R \cap L_{I,g} =^* R_{I,g}$ for all I and g. This defines $f : \omega \to \mathcal{P}(\mathbb{N})$ by $m \in f(n)$ iff $\langle n, m\rangle \in R$.

If $f \restriction I$ were not almost equal to f_I then we'd find infinitely many n with an m_n in $f(n) \mathbin{\triangle} f_I(n)$. But then $R \cap L_{I,g} \neq^* R_{I,g}$, where $g \in {}^\omega\omega$ follows $n \mapsto m_n$. □

2. No exact lifting is complete

Assume $\varphi : \mathbb{M} \to \mathcal{P}(\mathbb{N})/\mathit{fin}$ is an embedding and consider an exact lifting Φ of φ. The following proposition shows that Φ is not complete.

Proposition 2.1. *There is a sequence $\langle t_n : n \in \omega\rangle$ in $2^{<\omega}$ such that for the open set $O = \bigcup_{n\in\omega}\{n\}\times[t_n]$ we have $\Phi(O) \neq^* \Phi[O]$.*

Proof. Take, for each n, the monotone enumeration $\{k(n,i) : i \in \omega\}$ of $\Phi(n, \emptyset)$ and apply the equalities to find $t(n,i) \in 2^{i+2}$ such that $k(n,i) \in \Phi\bigl(n, t(n,i)\bigr)$. Use these $t(n,i)$ to define open sets $U_n = \bigcup_{i\in\omega}\{n\}\times\bigl[t(n,i)\bigr]$; observe that $\mu(U_n) \leqslant \sum_{i\in\omega} 2^{-i-2} = \frac{1}{2}$. It follows that $\Phi\bigl(\{n\}\times U_n^c\bigr)$ is infinite.

We let F be the closed set $\bigcup_{n\in\omega}\{n\}\times U_n^c$; its image $\Phi(F)$ meets every $\Phi(n,\emptyset)$ in an infinite set. For every n let i_n be the first index with $k(n,i_n) \in \Phi(F)$ and consider the open set $O = \bigcup_{n\in\omega}\{n\}\times\bigl[t(n,i_n)\bigr]$ and the infinite set $I = \bigl\{k(n,i_n) : n \in \omega\bigr\}$.

Observe the following

(1) $\Phi(O) \cap \Phi(F) =^* \emptyset$, because $O \cap F = \emptyset$;
(2) $I \subseteq \Phi(F)$, by our choice of the i_n; and
(3) $I \subseteq \Phi[O]$, by the choice of the $t(n,i_n)$.

It follows that $\bigl\langle t(n,i_n) : n \in \omega\bigr\rangle$ is as required. □

This completes the proof of the first half of the main argument.

3. OCA gives embeddings with exact liftings that are complete

We assume φ is an embedding of $\mathbb{M}$ into $\mathcal{P}(\mathbb{N})/\mathit{fin}$. We shall find, assuming OCA, an infinite set A and a lifting Φ of φ that is exact and complete when restricted to subsets of $A\times 2^{<\omega}$.

The infinite set A will come from an almost disjoint family on ω: we fix bijection π between ω and $2^{<\omega}$ and define for $x \in 2^\omega$ the set A_x by $A_x = \pi^{\leftarrow}\bigl[\{x\restriction n : n \in \omega\}\bigr]$.

For the rest of this section we fix an $\aleph_1$-sized subfamily $\mathcal{A}$ of the A_x's and enumerate it as $\{A_\alpha : \alpha < \omega_1\}$. Using OCA we shall show that all but countably many A_α are as required.

By construction the family $\mathcal{A}$ has a special property, commonly referred to as *neatness*; an almost disjoint family $\mathcal{C}$ is neat if there is a bijection π between ω

and $2^{<\omega}$ such that for every $C \in \mathcal{C}$ the set $\pi[C]$ is a subset of some branch x_C and, moreover, the map $C \mapsto x_C$ is one-to-one.

Our final lifting will be a limit, via Theorem 1.4, of a coherent family of liftings; these liftings will be defined only partially so we fix an exact lifting Ψ of φ to extend these partial liftings.

The key technical result is the following; we postpone its proof until Section 4.

Theorem 3.1 (OCA). *Let φ be an embedding of $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})/\mathit{fin}$ and let $\mathcal{A}$ be a neat almost disjoint family on $\mathbb{N}$ of size $\aleph_1$. Then for all but countably many $A \in \mathcal{A}$ there are $D \subseteq \mathbb{N}$ and a function $H : D \to A$ such that $\varphi(x) = H^{\leftarrow}[x]^*$ for all $x \subseteq A$.*

This theorem will now be applied to establish the following proposition.

Proposition 3.2 (OCA). *For every $f \in {}^\omega\omega$ there is a $\beta < \omega_1$ such that for every $\alpha \geqslant \beta$ there is a lifting $\Phi_{f,\alpha}$ of φ with $\Phi_{f,\alpha}(O) = \Phi_{f,\alpha}[O]$ whenever $O \subseteq B_{f,A_\alpha}$ and such that $\Phi_{f,\alpha}(n,s) \cap \Phi_{f,\alpha}(m,t) = \emptyset$ whenever $\{n\} \times [s]$ and $\{m\} \times [t]$ are disjoint.*

Proof. We fix $f \in {}^\omega\omega$ and show how to find β and $\Phi_{f,\alpha}$ for each $\alpha \geqslant \beta$. We transfer the almost disjoint family $\mathcal{A}$ to B_f by setting $C_\alpha = B_{f,A_\alpha}$ and $\mathcal{C} = \{C_\alpha : \alpha < \omega_1\}$.

It is fairly straightforward to show that $\mathcal{C}$ is neat; one stretches the bijection π to find an injection $\tilde{\pi}$ from B_f to $2^{<\omega}$ that maps every C_α onto a branch of $2^{<\omega}$ and different C_α to different branches.

This means that we can apply Theorem 3.1, to be proved in the next section, to the embedding φ_f of $\mathcal{P}(B_f)$ into $\mathcal{P}(\mathbb{N})/\mathit{fin}$, defined by $\varphi_f(O) = \varphi(O)$. This gives us a β and for every $\alpha \geqslant \beta$ a subset D_α of $\mathbb{N}$ and a function $H_\alpha : D_\alpha \to C_\alpha$ such that for every subset O of B_f the set $H_\alpha^{\leftarrow}[O]$ is a representative of $\varphi_f(O)$. We can define $\Phi_{f,\alpha}$ by $\Phi_{f,\alpha}(O) = H^{\leftarrow}[O]$ for $O \subseteq B_f$ and by setting $\Phi_{f,\alpha}(a) = \Psi(a)$ for the other elements of $\mathbb{M}$. $\square$

For each f we denote the minimum possible β by α_f.

Lemma 3.3. *If $f \leqslant^* g$ then $\alpha_f \leqslant \alpha_g$.*

Proof. Let $\alpha \geqslant \alpha_g$ and consider the lifting $\Phi_{g,\alpha}$. We define a lifting $\Phi_{f,\alpha}$ in a fairly obvious way: first fix m such that $f(n) \leqslant g(n)$ for $n \geqslant m$ and, if need be, redefine, for the duration of this proof, the values $\Psi(n,s)$ for $n < m$ and $s \in 2^{f(n)}$ so as to get $\Psi(n,s) \cap \Phi_{g,\alpha}(l,t) = \emptyset$ whenever this is needed.

Then, given $O \subseteq B_{f,A_\alpha}$ put $O_1 = \bigl\{\langle n,s\rangle \in O : n \geqslant m\bigr\}$ and put

$$U_O = \bigl\{\langle n,t\rangle \in B_{g,A_\alpha} : (\exists \langle n,s\rangle \in O_1)(s \subseteq t)\bigr\}.$$

We define, for $O \subseteq B_{f,A_\alpha}$,

$$\Phi_{f,\alpha}(O) = \Phi_{g,\alpha}[U_O] \cup \Psi\bigl[\bigl\{\langle n,s\rangle \in O : n < m\bigr\}\bigr].$$

Note that we implicitly defined $\Phi_{f,\alpha}(n,s) = \Phi\bigl[\bigl\{\langle n,t\rangle \in B_g : s \subseteq t\bigr\}\bigr]$ whenever $n \geqslant m$ and $\Phi_{f,\alpha}(n,s) = \Psi(n,s)$ when $n < m$. It follows that $\Phi_{f,\alpha}(O_1) = \Phi_{g,\alpha}[U_O] = \Phi_{f,\alpha}[O_1]$ and hence that

$$\Phi_{f,\alpha}(O) = \Phi_{f,\alpha}[O].$$

We already took care of the disjointness requirement so this $\Phi_{f,\alpha}$ witnesses that $\alpha \geqslant \alpha_f$ (once we use Ψ to define $\Phi_{f,\alpha}$ on the rest of $\mathbb{M}$). $\square$

We apply Lemma 1.2 to find α_∞ such that $\alpha_f \leqslant \alpha_\infty$ for all f.

We fix $\alpha \geqslant \alpha_\infty$ and put $A = A_\alpha$. For every $f \in \mathcal{F}$ we simply write Φ_f for the lifting $\Phi_{f,\alpha}$. For every f we extend Φ_f in a natural way to the set $C_f = \bigl\{\langle n,s\rangle :$

$n \in A, |s| \leqslant f(n)\bigr\}$: we demand that $\Phi_f(s,n) = \Phi_f(n,s^\frown 0) \cup \Phi_f(n,s^\frown 1)$ whenever appropriate; this makes Φ_f exact on C_f.

Lemma 3.4. *If $f \leqslant^* g$ then $\Phi_g \restriction C_f =^* \Phi_f$.*

Proof. Consider a potential sequence $\bigl\langle \langle n_i, s_i\rangle : i \in \omega\bigr\rangle$ of points in C_f where Φ_g and Φ_f disagree. By the disjointness condition and because the symmetric difference of $\Phi_f(n_i,s_i)$ and $\Phi_g(n_i,s_i)$ is always finite we can assume that $\Phi_f(n_i,s_i)$ does not meet $\Phi_g(n_j,s_j)$ when $i < j$. Let $O_g = \bigl\{\langle n_i, s\rangle \in B_g : i \in \omega, s_i \subseteq s\bigr\}$ and $O_f = \bigl\{\langle n_i, s\rangle \in B_f : i \in \omega, s_i \subseteq s\bigr\}$.

Observe that O_g and O_f determine the same open subset of $A \times 2^\omega$, so that $\Phi_g(O_g) =^* \Phi_f(O_g)$. It should be clear however that by the choice of the points $\langle n_i, s_i\rangle$ we have $\Phi_g[O_g] \neq^* \Phi_f[O_f]$, which is a contradiction. □

Observe that because $\mathfrak{b} = \aleph_2$ the family $\{C_f : f \in {}^\omega\omega\}$ is a $P_{\aleph_1}$-ideal on $A \times 2^{<\omega}$; we can therefore apply Theorem 1.4 to find one map Φ from $A \times 2^{<\omega}$ to $\mathcal{P}(\mathbb{N})$ such that $\Phi \restriction C_f =^* \Phi_f$ for all $f \in {}^\omega\omega$. This function Φ is almost as required.

First choose $m \in \omega$ and a $\leqslant^*$-cofinal family $\mathcal{F}$ consisting of increasing elements of ${}^\omega\omega$ such that $\Phi(n,s) = \Phi_f(n,s)$ whenever $f \in \mathcal{F}$, $n \geqslant m$ and $s \in 2^{f(n)}$. Without loss of generality the set $\{f(m) : f \in \mathcal{F}\}$ is unbounded — make m a bit larger if necessary (if no larger m can be found the family $\mathcal{F}$ is not even $\leqslant^*$-unbounded).

This immediately implies that Φ is exact on $(A \setminus m) \times 2^{<\omega}$; we simply modify Φ slightly on $(A \cap m) \times 2^{<\omega}$ to make it exact on the whole of $A \times 2^{<\omega}$. For all other elements a of $\mathbb{M}$ we put $\Phi(a) = \Psi(a)$.

The proof that Φ is complete is much like the proof of Lemma 3.3. Let $f \in {}^\omega\omega$ and $l \geqslant m$ such that $\Phi(n,s) = \Phi_f(n,s)$ whenever $n \leqslant l$. Given $O \subseteq B_f$ we first note that $\Phi(O) =^* \Phi_f(O)$, because both Φ and Φ_f are liftings.

To complete the proof we show that also $\Phi[O] =^* \Phi_f[O]$. Indeed, let $O' = \bigl\{\langle n,s\rangle \in O : n \geqslant l\bigr\}$, then $\Phi[O'] = \Phi_f[O']$, so we are left with showing $\Phi[O''] =^* \Phi_f[O'']$, where $O'' = O \setminus O'$. But O'' is finite so that by Lemma 1.1 we have $\Phi[O''] =^* \Phi(O'')$ and $\Phi_f(O'') =^* \Phi_f[O'']$; the equality $\Phi(O'') =^* \Phi_f(O'')$ holds because both maps are liftings.

4. Embedding $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})/\mathit{fin}$

In this section prove Theorem 3.1, thus completing the argument for our main result. We are given an an embedding φ of $\mathcal{P}(\mathbb{N})$ and a neat almost disjoint family $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$. We have to find an α such that for every $\beta \geqslant \alpha$ there are $D \subseteq \mathbb{N}$ and $H : D \to A_\alpha$ such that $\varphi(x) = H^{\leftarrow}[x]^*$ for all subsets x of A_α.

We begin by taking a lifting $\Phi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ of φ. We may assume, upon replacing $\Phi\bigl(\{n\}\bigr)$ by $\{n\} \cup \Phi\bigl(\{n\}\bigr) \setminus \bigcup_{i<n} \Phi\bigl(\{i\}\bigr)$, that the $\Phi\bigl(\{n\}\bigr)$ form a partition of $\mathbb{N}$. We shall identify $\mathbb{N}$ with $\mathbb{N} \times \mathbb{N}$ in such a way that $\Phi\bigl(\{n\}\bigr)$ corresponds to the vertical line $\{n\} \times \mathbb{N}$; we shall therefore write V_n for $\Phi\bigl(\{n\}\bigr)$.

For an $f \in {}^\omega\omega$ we write $L_f = \bigl\{\langle n,m\rangle : n \in \omega, m \leqslant f(n)\bigr\}$. The following lemma will be useful toward the end of the proof.

Lemma 4.1. *For each $a \subseteq \mathbb{N}$ there is $f \in {}^\omega\omega$ such that $\Phi(a) \setminus L_f = \bigcup_{n\in a} V_n \setminus L_f$.*

Proof. If $n \in a$ then $V_n \subseteq^* \Phi(a)$ and if $n \notin a$ then $V_n \cap \Phi(a) =^* \emptyset$; now let f code witnesses: if $n \in a$ then $V_n \setminus L_f \subseteq \Phi(a)$ and if $n \notin a$ then $V_n \setminus L_f \cap \Phi(a) = \emptyset$. □

We enumerate $\mathcal{A}$ as $\{A_\alpha : \alpha \in \omega_1\}$.

For $f \in {}^\omega\omega$ consider $\Phi_f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(L_f)$, defined by $\Phi_f(a) = \Phi(a) \cap L_f$ and observe that Φ_f induces a homomorphism from $\mathcal{P}(\mathbb{N})/\mathit{fin}$ to $\mathcal{P}(\mathbb{N})/\mathit{fin}$.

As in [4] OCA may now be applied to give us an $\alpha_f < \omega_1$ such that Φ_f is simple on A_α whenever $\alpha \geqslant \alpha_f$, where 'simple' means that there are $D \subseteq L_f$ and a finite-to-one function $h : D \to A_\alpha$ such that $\Phi_f(a) =^* h^\leftarrow[a]$ for all subsets a of A_α. As in the previous section we choose α_f as small as possible and we use the following lemma to fix α_∞ such that $\alpha_f \leqslant \alpha_\infty$ for all f.

Lemma 4.2. *If $f \leqslant^* g$ then $\alpha_f \leqslant \alpha_g$.*

Proof. Take $\alpha \geqslant \alpha_g$ and fix $D \subseteq L_g$ and $h : D \to A_\alpha$ such that $\Phi_g(a) =^* h^\leftarrow[a]$ for all $a \subseteq A_\alpha$. Now simply let $D_1 = D \cap L_f$ and $h_1 = h \restriction D_1$; clearly $\Phi_f(a) =^* \Phi_g \cap L_f =^* h^\leftarrow[a] \cap L_f = h_1^\leftarrow[a]$ for all $a \subseteq A_\alpha$. We see that $\alpha \geqslant \alpha_f$. ☐

For the rest of the proof we fix an $\alpha \geqslant \alpha_\infty$ and show that $A = A_\alpha$ is as required. For each $f \in {}^\omega\omega$ we take D_f and $h_f : D_f \to A$ as above. We intend to find D and H by an application of Theorem 1.3.

Lemma 4.3. *If $f \leqslant g$ then $D_f =^* D_g \cap L_f$ and $h_g \restriction D_f =^* h_f$.*

Proof. The first equality is clear: by construction $D_g =^* \Phi_g(A)$ and $D_f =^* \Phi_f(A)$, so that $D_g \cap L_f =^* \Phi_g(A) \cap L_f =^* \Phi_f(A) =^* D_f$.

To prove the second equality let x be an infinite subset of $D_g \cap D_f$ such that $h_f(i) \neq h_g(i)$ for all $i \in x$; because h_f and h_g are finite-to-one we can assume that $h_f[x] \cap h_g[x] = \emptyset$. But then we would have a contradiction because on the one hand $\Phi_g\bigl(h_f[x]\bigr) \cap \Phi_g\bigl(h_g[x]\bigr) =^* h_g^\leftarrow\bigl[h_f[x]\bigr] \cap h_g^\leftarrow\bigl[h_g[x]\bigr] = \emptyset$ while on the other hand $x \subseteq h_f^\leftarrow\bigl[h_f[x]\bigr] \cap h_g^\leftarrow\bigl[h_g[x]\bigr] \subseteq^* \Phi_g\bigl(h_f[x]\bigr) \cap \Phi_g\bigl(h_g[x]\bigr)$. ☐

We apply Theorem 1.3 to the family $\{F_f : f \in {}^\omega\omega\}$ of functions defined by $F_f(p) = \bigl\langle \chi_{D_f}(p), h_f(p)\bigr\rangle$ to find a function $F : \omega \times \omega \to 2 \times \omega$ that uniformizes this family. We set $D = \{p : F_1(p) = 1\}$ and $H = F_2 \restriction D$.

For the rest of the proof the letters n and m will refer to elements of A.

Lemma 4.4. *There are only finitely many pairs $\langle n, m\rangle$ for which $H^\leftarrow(n) \cap V_m$ is infinite.*

Proof. Assume we have $\bigl\{\langle n_i, m_i\rangle : i \in \omega\bigr\}$ with $H^\leftarrow(n_i) \cap V_{m_i}$ infinite for all i and $n_i, m_i < n_j, m_j$ whenever $i < j$. Choose $f \in {}^\omega\omega$ as per Lemma 4.1 for the sets $b = \{n_i : i \in \omega\}$ and $c = \{m_i : i \in \omega\}$ and choose $g > f$ such that $H^\leftarrow(n_i) \cap V_{m_i} \cap (L_g \setminus L_f) \neq \emptyset$ for all i.

We obtain a contradiction as before: b and c are disjoint, hence $\Phi(b)$ and $\Phi(c)$ are almost disjoint. On the other hand $h_g^\leftarrow[b] \setminus L_f \subseteq^* \Phi(b)$ and $\bigcup_{m \in c}\{m\} \times \bigl(f(m), g(m)\bigr] \subseteq \Phi(c)$; the intersection of the smaller sets is infinite. ☐

Lemma 4.5. *There are only finitely many m in A for which the set $\{n \in A : H^\leftarrow(n) \cap V_m \neq \emptyset\}$ is infinite.*

Proof. Let b be the set of m in A for which $I_m = \{n \in A : H^\leftarrow(n) \cap V_m \neq \emptyset\}$ is infinite. Thin out b to get $I_m \setminus b$ infinite for all m in b. Choose $f \in {}^\omega\omega$ as per Lemma 4.1 for A and b, so that $\Phi(b) \setminus L_f = (b \times \omega) \setminus L_f$ and $\Phi(A \setminus b) \setminus L_f = \bigl((A \setminus b) \times \omega\bigr) \setminus L_f$.

Because the sets $H^\leftarrow(n)$ are pairwise disjoint we can a one-to-one choice function $m \mapsto n_m$ for the family $\{I_m \setminus b : m \in b\}$ such that $H^\leftarrow(n_m) \cap V_m \setminus L_f \neq \emptyset$ for all m.

We choose a function $g > f$ that captures these intersections: $H^{\leftarrow}(n_m) \cap V_n \cap (L_g \setminus L_f) \neq \emptyset$ for all m. It follows that $H^{\leftarrow}[b] \cap (L_g \setminus L_f)$ meets $(A \setminus b) \times \omega$ in an infinite set. However, by the choice of f and the properties of h_g the set $H^{\leftarrow}[b] \cap (L_g \setminus L_f)$ is almost equal to $\bigcup_{m \in b} \{m\} \times \bigl(f(m), g(m)\bigr]$ which is disjoint from $(A \setminus b) \times \omega$. $\square$

Lemma 4.6. *For every $n \in A$ the set $\{m \in A : H^{\leftarrow}(n) \cap V_m \neq \emptyset\}$ is finite.*

Proof. Fix $f \in {}^{\omega}\omega$ such that for all n and m in A: *if* $n \neq m$ and $H^{\leftarrow}(n) \cap V_m \neq \emptyset$ *then* $H^{\leftarrow}(n) \cap V_m \cap L_f \neq \emptyset$. Now note that $H^{\leftarrow}(n) \cap L_f =^* h_f^{\leftarrow}(n)$, so that $H^{\leftarrow}(n) \cap L_f$ is finite. $\square$

Putting these lemmas together we see that there are M and N in ω such that $N \geqslant M$ and

(1) if $n, m \geqslant M$ and $n \neq m$ then $H^{\leftarrow}(n) \cap V_m$ is finite;
(2) if $m \geqslant M$ then V_m meets only finitely many $H^{\leftarrow}(n)$; and
(3) if $n < M$ and $m \geqslant N$ then $H^{\leftarrow}(n) \cap V_m = \emptyset$.

(Note that M should be chosen first, to ensure 1 and 2.)

By 1 and 2 we can fix $h \in {}^{\omega}\omega$ such that $H^{\leftarrow}(n) \cap V_m \subseteq L_h$ whenever $n, m \geqslant M$ and $n \neq m$; an application of 3 then tells us that $H^{\leftarrow}(n) \setminus L_h = V_n \setminus L_h$ for $n \geqslant N$. We also redefine H on the set $N \times \omega$ to get $H^{\leftarrow}(n) = V_n$ for $n < N$.

Now let $b \subseteq A$ and fix $f > h$ as per Lemma 4.1. By the choice of f and h we have

$$\Phi(b) \setminus L_f = (b \times \omega) \setminus L_f = H^{\leftarrow}[b] \setminus L_f.$$

The redefined H still satisfies $H \restriction L_f =^* h_f$ so that

$$\Phi(b) \cap L_f =^* h_f^{\leftarrow}[b] =^* H^{\leftarrow}[b] \cap L_f.$$

This shows that H is as required.

5. Comments and questions

In this section we comment on two earlier results about the embeddability of $\mathbb{M}$ into $\mathcal{P}(\mathbb{N})/\mathit{fin}$ and raise some questions.

5.1. **Martin's Axiom does not imply embeddability.** A consequence of our result is that Martin's Axiom does not imply that $\mathbb{M}$ can be embedded into $\mathcal{P}(\mathbb{N})/\mathit{fin}$; this is so because the conjunction $\mathsf{OCA} + \mathsf{MA}$ is consistent — it follows from the Proper Forcing Axiom and it can be proved consistent in an ω_2-length iterated forcing construction.

In [7] Frankiewicz and Gutek assert that MA implies there is a measure-preserving embedding φ of $\mathbb{M}$ into $\mathcal{P}(\mathbb{N})/\mathit{fin}$ — measure preserving in the sense that for every element a one has $\mu(a) = d\bigl(\varphi(a)\bigr)$. Here d is the *asymptotic density*, defined by

$$d(X) = \lim_{n \to \infty} \frac{\bigl|X \cap \{1, 2, \ldots, n\}\bigr|}{n},$$

for those subsets X of $\mathbb{N}$ for which the limit exists. Of course for this to make sense we must consider the standard incarnation of $\mathbb{M}$ as the quotient of the Borel algebra of the unit interval by the ideal of measure-zero sets.

The reader is likely to be interested where the argument has a gap. It seems that the principal error in their proof is in the following lemma, which is the key step in the construction of the embedding.

Lemma 5.1 (MA). *If $\mathbb{L}$ is a subalgebra of $\mathbb{M}$ of size less than $\mathfrak{c}$ and if $a \in \mathbb{M} \setminus \mathbb{L}$ then every measure-preserving embedding φ of $\mathbb{L}$ into $\mathcal{P}(\mathbb{N})/\mathit{fin}$ can be extended to a measure-preserving embedding ψ of the algebra generated by $\mathbb{L} \cup \{a\}$ into $\mathcal{P}(\mathbb{N})/\mathit{fin}$.*

It is relatively easy to see that this lemma is true, in ZFC, for countable $\mathbb{L}$ — indeed, a value $\psi(a)$ is readily constructed by recursion. This lemma is false for $\mathbb{L}$ of size $\aleph_1$, as can be seen from the following example. We work on the interval $(0, 1]$. We split $(0, 1]$ into intervals $\{I_n : n \in \mathbb{N}\}$, where $I_{2n-1} = (2^{-(n+1)}, 2^{-n}]$ and $I_{2n} = I_{2n-1} + \frac{1}{2}$. Next let $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ be an $\aleph_1$-sized almost disjoint family, where A_α consists of even/odd numbers whenever α is even/odd. For $\alpha < \omega_1$ we put $a_\alpha = \bigcup_{n \in A_\alpha} I_n$; note that $\{a_\alpha : \alpha < \omega_1\}$ is an independent family in $\mathbb{M}$ in that no element is in the algebra generated by the other elements of the family — we let $\mathbb{L}$ be the subalgebra of $\mathbb{M}$ generated by this family. Observe that $a = (0, \frac{1}{2}]$ does not belong to $\mathbb{L}$.

Let $\mathcal{L} = \{L_\alpha : \alpha < \omega_1\}$ be a Luzin-type almost disjoint family, which means that for no two disjoint uncountable subsets S and T of ω_1 one can find a subset X of $\mathbb{N}$ such that $L_\alpha \subseteq^* X$ for $\alpha \in S$ and $L_\alpha \cap X =^* \emptyset$ for $\alpha \in T$ — see [10] or [16, Theorem 4.1] . We assume the family $\mathcal{L}$ lives on the set $Z = \{n! : n \in \mathbb{N}\}$ — note that $d(Z) = 0$.

We can apply the countable version of Lemma 5.1 to find a measure-preserving embedding from the algebra $\mathbb{L}$ into $\mathcal{P}(\mathbb{N} \setminus Z)/\mathit{fin}$. Now augment this embedding to obtain an embedding φ of $\mathbb{L}$ into $\mathcal{P}(\mathbb{N})/\mathit{fin}$ such that $\varphi(a_\alpha) \cap Z = L_\alpha$ for all α. There is no way to extend φ to an embedding of $\mathbb{L} \cup \{a\}$ into $\mathcal{P}(\mathbb{N})/\mathit{fin}$, measure-preserving or otherwise: because $a_\alpha < a$ if α is odd and $a_\alpha \wedge a = 0$ if α is even we should have $L_\alpha \subseteq^* \psi(a)$ for odd α and $L_\alpha \cap \psi(a) =^* \emptyset$ for even α, which is impossible by the choice of $\mathcal{L}$.

5.2. **The oracle-c.c. method.** In [6] Frankiewicz sketched a proof of the consistency of the nonembeddability of $\mathbb{M}$, using Shelah's oracle-c.c. method. Certainly this sketch is quite incomplete but it also appears to be fundamentally erroneous. Before we can point out where Frankiewicz went wrong we give a rough outline of the oracle-c.c. method.

Shelah's oracle-c.c. method — see [3], [12, Chapter IV] and [14, Chapter IV] — provides us with a way of preventing that various undesirable objects will appear once an iterated forcing construction is underway. It would take us too far afield to explain this method in full, suffice it to say that one builds c.c.c. partial orders of size $\aleph_1$ whose antichains are kept under tight control by a $\diamondsuit$-sequence (an oracle).

A consistency proof, along these lines, for the nonembeddability of $\mathbb{M}$ would run as follows: given an embedding $\varphi : \mathbb{M} \to \mathcal{P}(\mathbb{N})/\mathit{fin}$, construct a partial order $\mathbb{P}$ and a $\mathbb{P}$-name $\dot{X}$ for a Borel set such that there is *no* $\mathbb{P} * \mathrm{Fn}(\omega, 2)$-name $\dot{A}$ for a subset of $\mathbb{N}$ that satisfies the following two requirements for every Borel set Y from the ground model:

$$\Vdash \text{“} Y \subseteq \dot{X} \Rightarrow \varphi(Y) \subseteq^* \dot{A} \text{”} \text{ and } \Vdash \text{“} \dot{X} \subseteq Y \Rightarrow \dot{A} \subseteq^* \varphi(Y) \text{”}$$

The extra factor $\mathrm{Fn}(\omega, 2)$ is a necessary technical device (implicit in any finite-support iteration) that will enable one to improve a given $\diamondsuit$-sequence into an oracle with the property that if the rest of the iteration is kept under its control one will never encounter an undesirable name $\dot{A}$ as above. This, together with a reflection argument involving $\diamondsuit$ on ω_2, will ensure that ultimately no embedding of $\mathbb{M}$ into $\mathcal{P}(\mathbb{N})/\mathit{fin}$ will remain.

Frankiewicz' sketch is based on Shelah's proof, from [13], that it is consistent that the natural quotient homomorphism q from $\mathrm{Bor}(\mathbb{C})$ onto $\mathbb{M}$ does not split, i.e., there is no homomorphism $h : \mathbb{M} \to \mathrm{Bor}(\mathbb{C})$ such that $q \circ h = \mathrm{Id}_{\mathbb{M}}$ (such a homomorphism is also called a lifting of $\mathbb{M}$ into $\mathrm{Bor}(\mathbb{C})$).

The sketch consists basically of two parts.

(1) A copy of Shelah's construction of the partial order from [13] with some (questionable) modifications, and
(2) the unsupported assertion that "From now on the proof goes exactly as in Shelah [13]".

The problem with the second part is that Shelah's argument was written up in such a way that it would also apply to the category algebra; this is the quotient algebra $\mathbb{K} = \mathrm{Bor}(\mathbb{C})/\mathcal{M}$, where $\mathcal{M}$ is the ideal of meager sets. As none of the modifications is $\mathbb{M}$-specific Frankiewicz' sketch would also lead to a proof that it is consistent for there to be no embedding of the category algebra into $\mathcal{P}(\mathbb{N})/\mathit{fin}$. This, however, is impossible: the category algebra is known to be embeddable into $\mathcal{P}(\mathbb{N})/\mathit{fin}$. This is most readily seen by noting its Stone space is separable and hence a continuous image of $\beta\mathbb{N} \setminus \mathbb{N}$.

One very problematic modification occurs almost at the beginning: we are given a countable partial order $\mathbb{P}$ and a $\mathbb{P}*\mathrm{Fn}(\omega,2)$-name $\dot{A}$ for a subset of $\mathbb{N}$. We are then promised an extension $\mathbb{P}'$ of $\mathbb{P}$ and an infinite subset B of $\mathbb{N}$ such that some condition in $\mathbb{P}' * \mathrm{Fn}(\omega,2)$ will force either $B \subseteq \dot{A}$ or $B \cap \dot{A} = \emptyset$. This is patently impossible if $\dot{A}$ happens to be the name of the generic subset that is added by $\mathrm{Fn}(\omega,2)$: it is well-known that this set and its complement meet every infinite subset of $\mathbb{N}$ from the ground model.

5.3. **Some questions.** It would be of interest to know whether there are other ways of proving the nonembeddability of the measure algebra consistent. To be specific we ask.

Question 5.1. Can the consistency of the nonembeddability of $\mathbb{M}$ be established by the oracle-c.c. method?

Because the oracle-c.c. method relies on having $\diamondsuit$ in every intermediate model it seems to produce models with $\mathfrak{c} \leqslant \aleph_2$ only; the known models for OCA satisfy $\mathfrak{c} = \aleph_2$ as well. Therefore the following question 'really' asks for a new method.

Question 5.2. Is the nonembeddability of $\mathbb{M}$ consistent with larger values for $\mathfrak{c}$?

Many statements that were proved consistent by the oracle-c.c. method were later shown to follow from OCA (or $\mathsf{OCA} + \mathsf{MA}$), see, e.g., [8]. One of the major questions that remains is.

Question 5.3. Does OCA or even PFA imply that the quotient homomorphism from $\mathrm{Bor}(\mathbb{C})$ onto $\mathbb{M}$ (or onto $\mathbb{K}$) does not split?

Another question is to delineate what the subalgebras of $\mathcal{P}(\mathbb{N})/\mathit{fin}$ (or, dually, the zero-dimensional continuous images of $\beta\mathbb{N} \setminus \mathbb{N}$) look like. The ZFC results can, broadly, be divided into two categories: the first contains the easy result that every separable compact space is a continuous image of $\beta\mathbb{N} \setminus \mathbb{N}$; the second contains moderately difficult constructions of embeddings into $\mathcal{P}(\mathbb{N})/\mathit{fin}$ of assorted algebras — these are generally small in some sense or another, so that the construction can be seen to terminate. The smallness conditions all seem to imply that the algebra in question is incomplete. This suggests that the following question might have a positive answer.

Question 5.4. Is it consistent that every complete Boolean algebra that is embeddable into $\mathcal{P}(\mathbb{N})/\mathit{fin}$ must be σ-centered?

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