Greedy Algorithms

- At each step in the algorithm, one of several choices can be made.
- **Greedy Strategy**: make the choice that is the best at the moment.
- After making a choice, we are left with **one subproblem** to solve.
- The solution is created by making a sequence of **locally optimal** choices.

A greedy algorithm does not always achieve a globally optimal solution.
But even when the final solution is not optimal:

"Greedy, for lack of a better solution, is good."

Greedy Algorithms: Optimality Conditions

Greedy Choice property:
A globally optimal solution can be arrived at by making a locally optimal (greedy) choice.

Optimal Substructure:
An optimal solution to the problem contains within it optimal solutions to subproblems.

Greedy Example: Minimum Spanning Trees

- A TV cable company wants to connect a set of N buildings such that the total amount of cable is minimized.
- Interconnect N pins in an electronic circuit using the least amount of wire.
- Create a highway infrastructure among N cities that minimizes the total length, such that every city is reachable from any other city.
- *and many others ...*

Minimum Spanning Trees

Problem:

Given: a connected, undirected graph $G = (V, E)$, where each edge (u, v) has a weight $w(u, v)$.

Find: a tree $T \subseteq E$ that connects all the vertices in V such that it has a minimum total weight $w(T) = \sum_{(u,v) \in T} w(u, v)$.

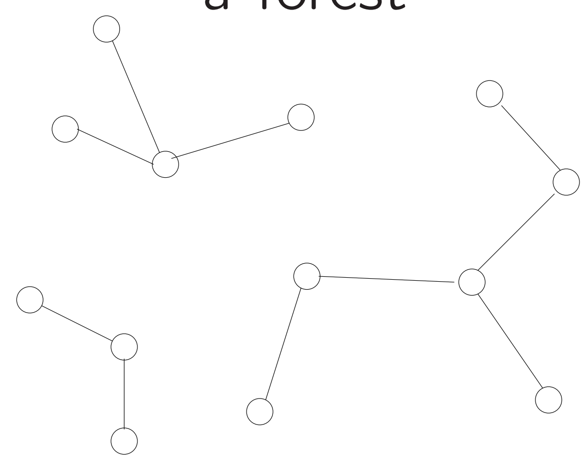
Trees and Forests

Tree: A tree is a connected, acyclic, undirected graph.

Forest: If an undirected graph is acyclic, but possibly disconnected, is it a forest.

a tree

a forest



Properties of Trees (Appendix B.5)

If $G = (V, E)$ be an undirected graph, the following statements are equivalent:

1 G is a tree;

2 Any two vertices in G are connected by a unique simple path;

3 G is connected, but if any edge is removed from E, the resulting graph is disconnected;

4 G is connected, and $|E| = |V|$ - 1;

5 G is acyclic, and $|E| = |V|$ - 1;

6 G is acyclic, but if any edge is added to E, the resulting graph contains a cycle.

Minimum Spanning Trees

- Definition: Let $G(V, E)$ be any undirected graph, $T(V, E')$ is said to be a **spanning tree** of $G(V, E)$ if $E' \subseteq E$ and $T(V, E')$ is a tree.

- **Problem**: given a connected, undirected weighted graph, find a *spanning tree* using edges that minimize the total weight.

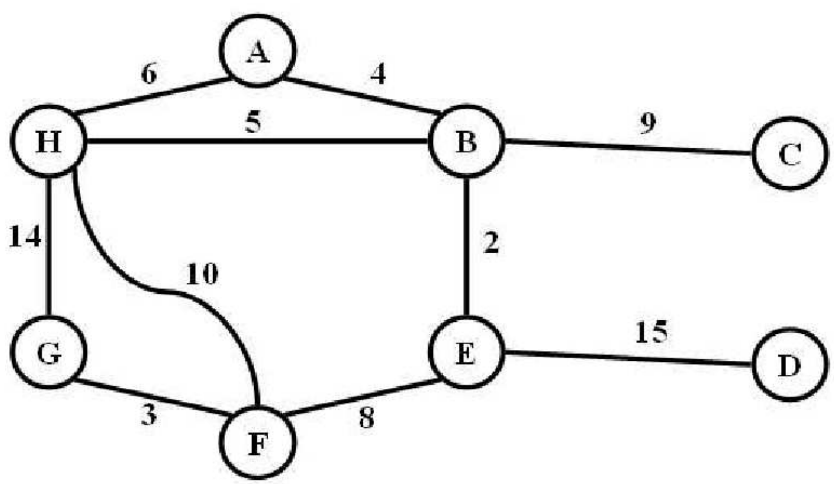
- The weight of a spanning tree is the sum of the edge weights.

- **Input**: An undirected graph $G(V, E)$ where each edge has a weight associated.

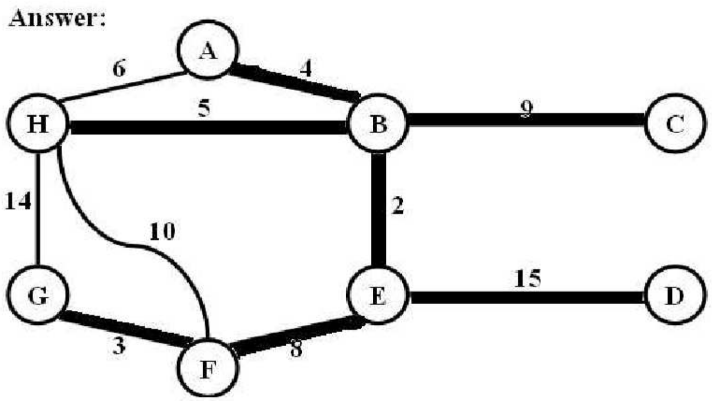
- **Output**: A minimum weight spanning tree of G.

An Example

Which edges form the minimum spanning tree (MST) of the graph below?

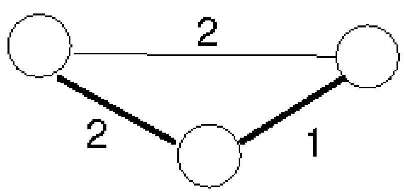


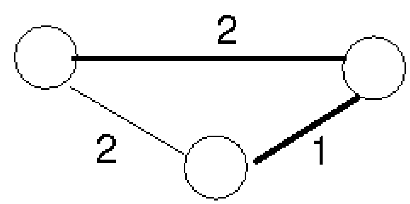
An Answer



Is the MST of a graph unique?

No, a graph can have more than one MSTs!





Optimal Substructure

MSTs satisfy the **optimal substructure** property: **an optimal (minimum spanning) tree is composed of optimal (MS) subtrees**.

- Let T be an MST of G, and an edge $(u, v) \in T$.
- Removing (u, v) partitions T into two trees T_1 and T_2.
- Claim: T_1 is an MST of $G_1 = (V_1, E_1)$ and T_2 is an MST of $G_2 = (V_2, E_2)$. (**Do V_1 and V_2 share vertices? why?**)
- Proof (*cut and paste*): $w(T) = w(u, v) + w(T_1) + w(T_2)$ (there cannot be a better tree than T_1 or T_2, otherwise, using *cut and paste*, we would get a spanning tree T' with smaller total weight than T)

Idea of solving the MST problem: grow a MST

General Idea: Grow a minimum spanning tree – prior to each iteration, keep A as a subset of edges from a minimum spanning tree.

```
Generic-MST (G, w)
    A := ∅
    while A does not form a spanning tree
        find an edge (u, v) that is safe for A;
        A := A ∪ (u, v));

    return A;
```

safe means $A \cup \{(u, v)\}$ is also a subset of certain MST

What kind of edges are safe?

Definitions:

- A **cut** $(S, V - S)$ of an undirected graph $G = (V, E)$ is a partition of V.
- An edge $(u, v) \in E$ **crosses** the cut $(S, V - S)$ if $u \in S$ and $v \in V - S$, or vice versa.
- A cut **respects** a set A of edges if no edge in A crosses the cut.
- An edge is a **light edge** crossing a cut if its weight is the minimum of any edge crossing the cut.

What kind of edges are safe?

Theorem 23.1

- Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E.
- Let T be a MST of G, and let A be a subset of edges s.t. $A \subseteq T$.
- Let $(S, V - S)$ be a cut of G that respects A.
- Let (u, v) be a light edge crossing the cut $(S, V - S)$.
- Then (u, v) is safe for A (i.e., $A \cup (u, v)$ will be a subset of a MST).

Theorem 23.1

Proof: Let T be a MST that includes A, and assume that T does not contain the min-weight edge (u,v), since if it does, we are done.

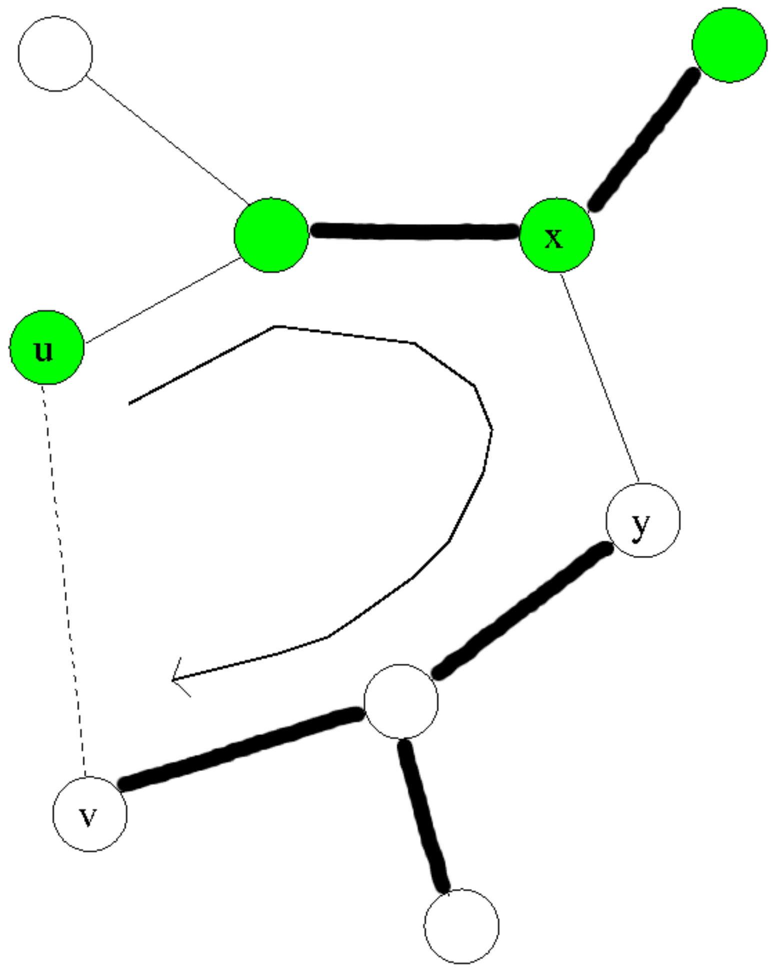
1. Construct another MST T' that includes $A \cup \{(u,v)\}$.
   $T' = T - \{(x,y)\} \cup \{u,v\}$ (Figure 23.3)

2. $w(T') = w(T) - w(x,y) + w(u,v) \leq w(T)$. But T is a MST, so $w(T) \leq w(T')$; thus, T' must be a MST also.

3. Since $A \subseteq T$ and $(x,y) \notin A \Rightarrow A \subseteq T'$; thus $A \cup \{(u,v)\} \subset T'$

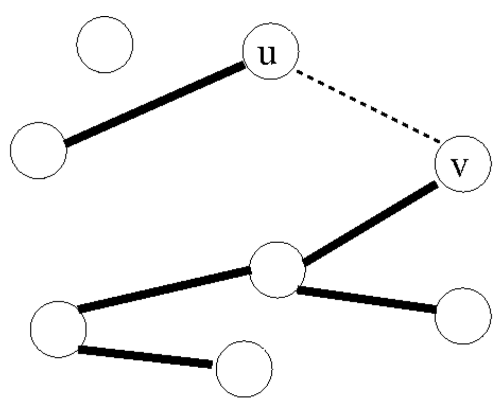
Since T' is a MST, (u,v) is safe for A.

Figure 23.3



Corollary 23.2

- Let A be a subset of an MST.
- Let $G_A = (V, A)$ be the forest induced by A.
- Let $C = (V_C, E_C)$ be a tree in the forest G_A.
- If (u, v) is a light edge connecting C to some other tree in A, then (u, v) is safe for A (i.e, $A \cup (u, v)$ will be a subset of a MST) .



Two algorithms for MST: Two different schemes of maintaining A

Based on different approaches of maintaining A, we have two algorithms: **Kruskal's algorithm** and **Prim's algorithm**:

- Kruskal's algorithm keeps the set A as a forest (a set of disjoint sets).
- Prim's algorithm grows a single tree A.

Kruskal's Algorithm

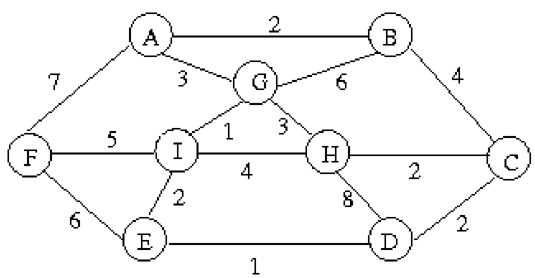
Basic idea: To grow a sparse forest A into a tree.

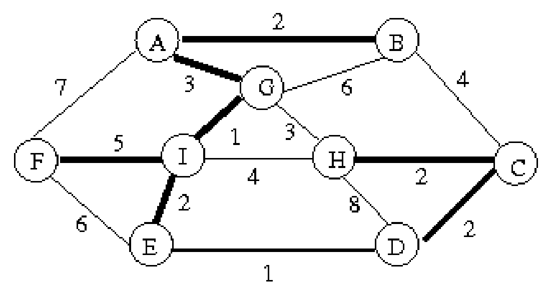
- At the beginning, each vertex is considered to be a different tree. A is the forest containing those trees.
- **Grow this forest into a tree**
  - Sort the edges in nondecreasing order by weight and put them in a list L.
  - For each edge in L, in order:
    - Remove the first edge (u, v) from L (i.e. the cheapest edge);
    - If (u, v) connects two trees (i.e., T_i and T_j) without introducing any cycle, then grow T_i and T_j into a bigger tree; otherwise discard (u, v).

  $>$ (u, v) *is* **safe** *for* A *by Corollary 23.2.*

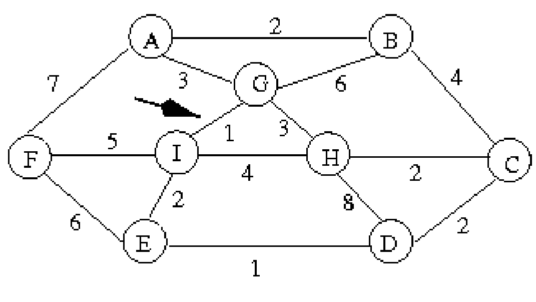
An example

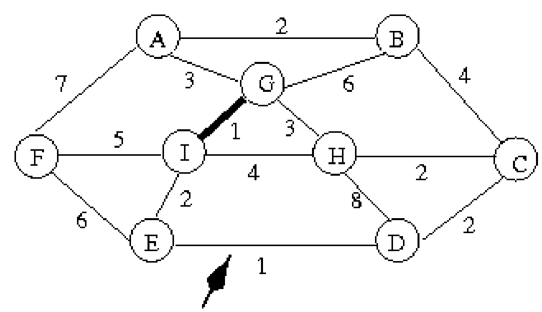
original graph



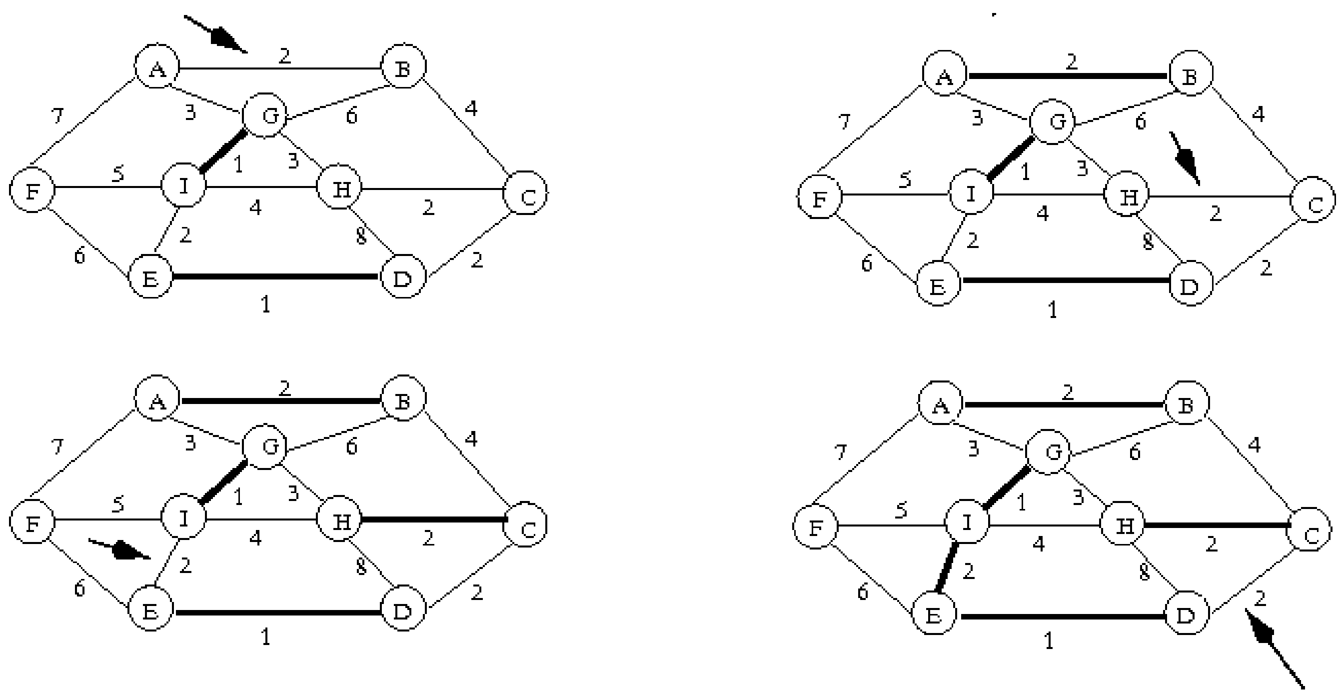


MST
w(T)= 18

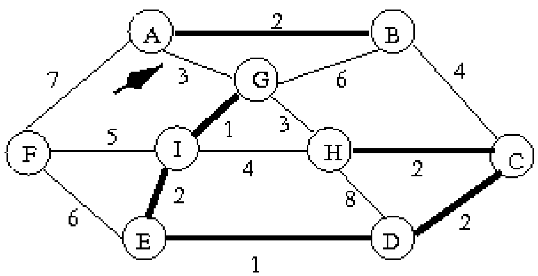


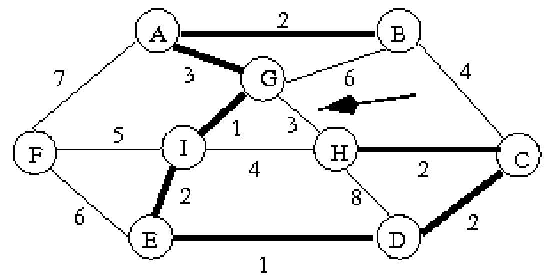


An example, Cont'd

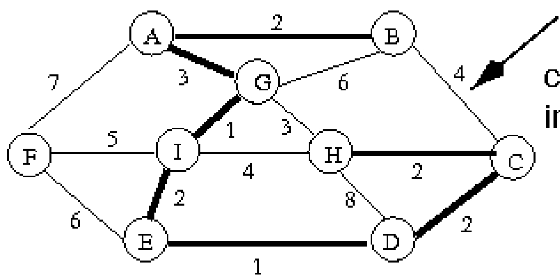


An example, Cont'd

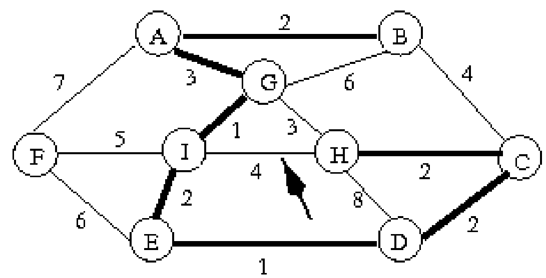




cycle will be introduced

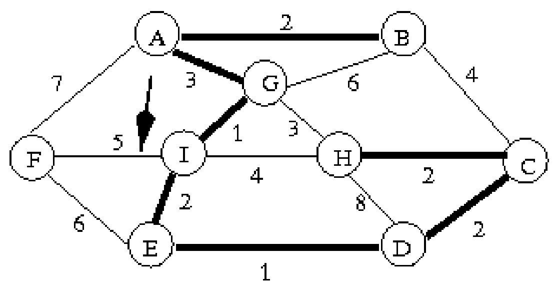


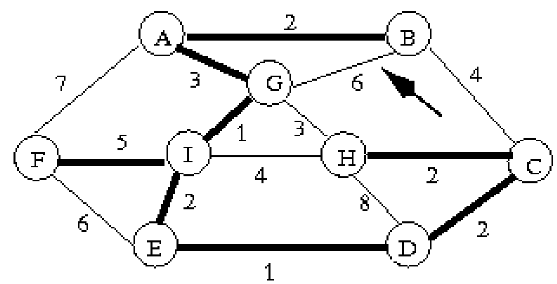
cycle will be introduced



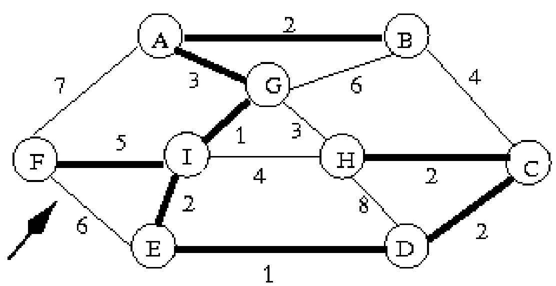
cycle will be introduced

An example, Cont'd

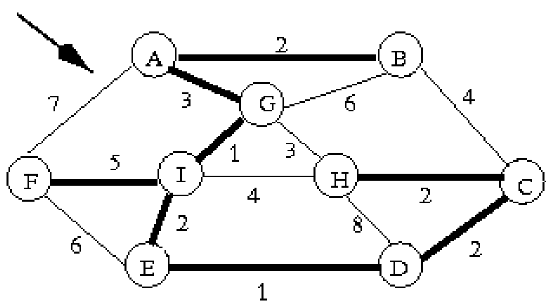




cycle will be introduced



cycle will be introduced



cycle will be introduced

Kruskal's algorithm: Implement a Forest

Q: How to implement a forest?
A: use a disjoint-set data structure to maintain several disjoint sets of elements. Each set represents a tree.

Q: How to check if a cycle is formed?
A: Each set/tree has a set/tree ID (unique representative). When you try to connect two vertices in the same tree (with the same tree ID), a cycle will be formed.

A simple data structure for Forests (Disjoint sets)

The operations we need to support:

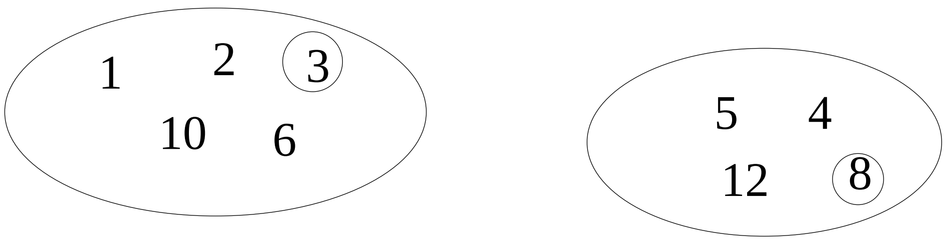
- Find-Set (return the set/tree ID).
- Union (combine two sets/trees into one larger set/tree).
- Make-Set (construct set/tree).

A simple solution: linked lists:

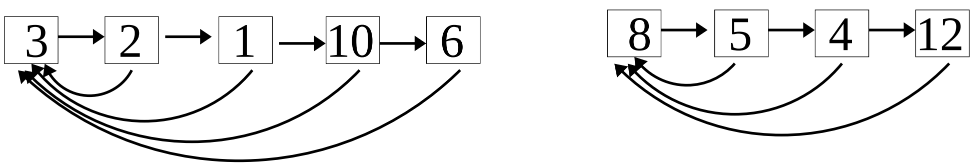
- Maintain elements in same set as a linked list with each element having a pointer to the first element of the list (unique representative).
- Each list maintains pointers *head* to the representative, and *tail* to the last object in the list.

Disjoint Sets: Implementation

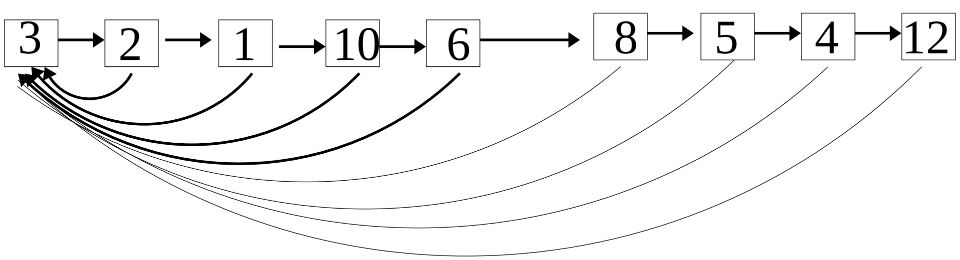
Sets



Representation



Union-Set



Disjoint Sets: Time complexity

- Make-Set(v): make a list with one element
  $\Rightarrow O(1)$ time.
- Find-Set(u): follow pointer and return the unique representative
  $\Rightarrow O(1)$ time.
- Union(u, v): point all the pointers of v's elements to u's unique representative
  $\Rightarrow O(|v|)$ time.
  $\Rightarrow |V|$ Union operations can take $\Theta(|V|^2)$ time.
  Can do better, using the **weighted union heuristic**.

Disjoint Sets: Weighted union heuristic

Augment the representation:

- Store the length of the list with each list.
- Always append the smaller list onto the longer list.

Theorem 21.1
Using the linked-list representation of disjoint sets and the weighted-union heuristic, a sequence of m Make-Set, Union, and Find-Set operations, n of which are Make-Set operations, takes $O(m + n \lg n)$ time.

Kruskal's Algorithm

MST-Kruskal (G, w)

A := ∅;
for each vertex $v \in V$
    Make-Set(v);   /* construct trees */

Sort the edges of E by weight ;

for each edge $(u, v) \in E$, in oder
    if Find-Set(u) $\neq$ Find-Set(v)   /* Not in the same tree */
        $A := A \cup \{(u, v)\}$
        Union-Set(u, v);   /* combine two trees into one */

Running time for Kruskal's algorithm

1. Sort: $O(|E|lg|E|)$.
2. $|V|$ Make-Set calls.
3. $2|E|$ Find-Set() calls.
4. $|V|-1$ Union-Set calls.

   Total: $2|E|+2|V|-1$ operations on the disjoint sets, $|V|$ of each are Make-Set operations
   $\Rightarrow O(2|E|+2|V|-1+|V|\lg|V|)$ time complexity, by Theorem 21.1.

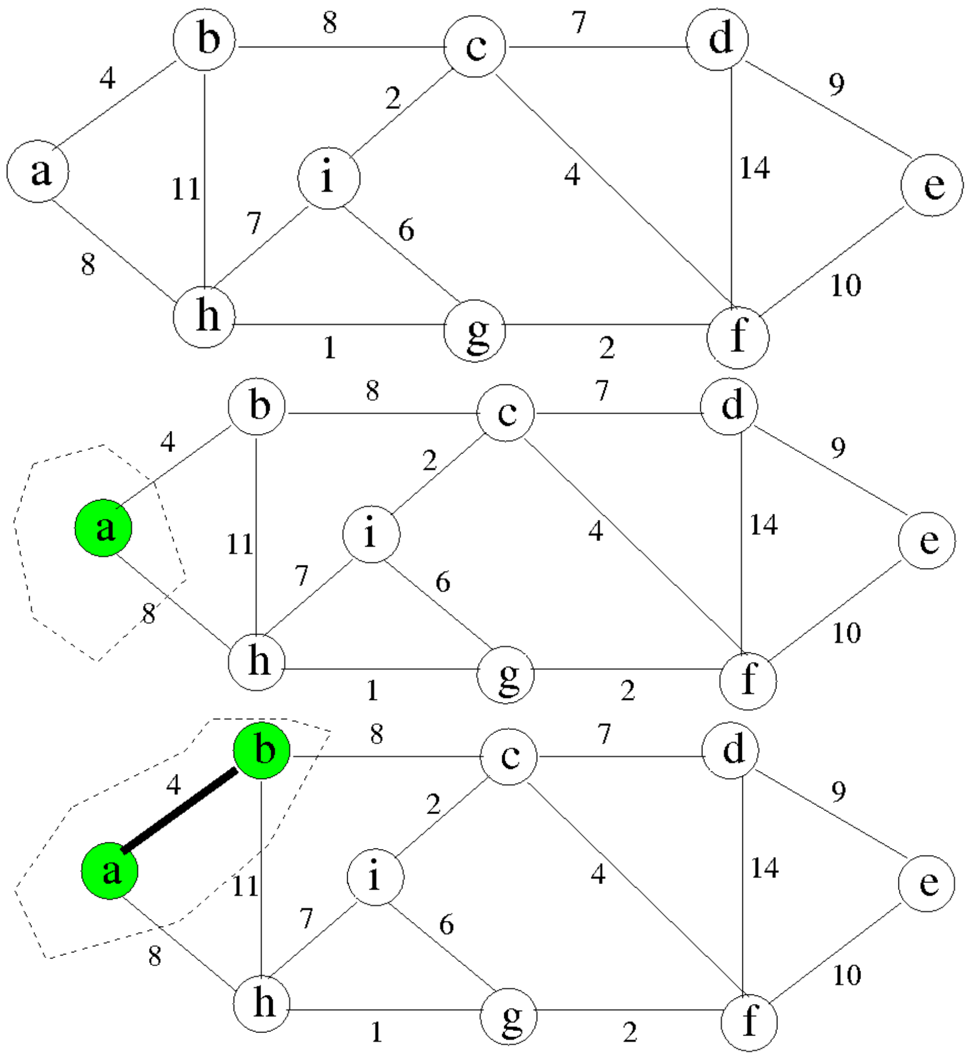
Overall, the complexity for Kruskal's algorithm is:
$O(|E|lg|E|) = O(|E|lg|V|)$.

Prim's algorithm

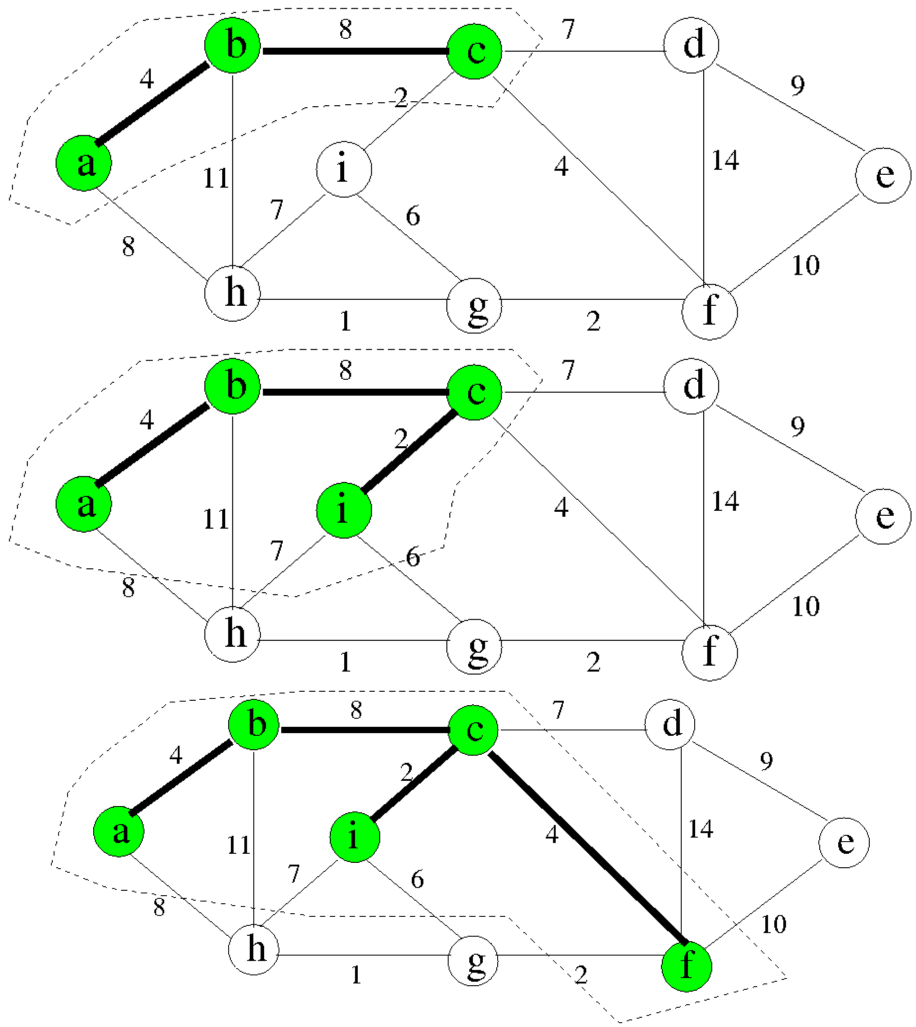
Basic idea: To grow a single tree A (from a one-node tree to a MST).

- Select an arbitrary vertex to start the tree A; Let V_A be the vertices covered by A.
- **growing the tree** A:
  - each time select an edge (u, v) of minimum weight connecting a vertex in V_A and a vertex outside of V_A.
  - include (u, v) into A.

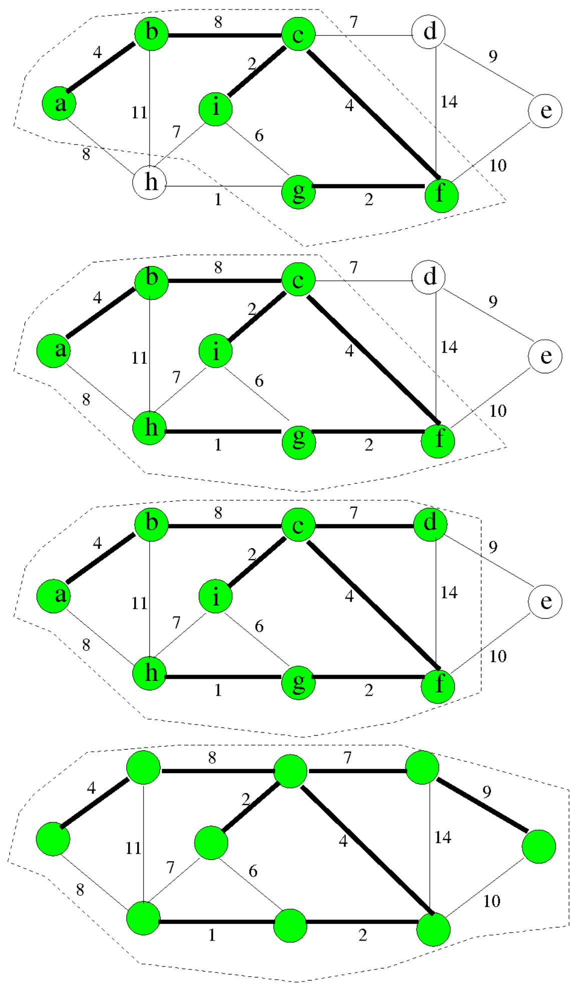
An example


a b c d e f g h i
4 8 7 9 14 10 2 1 6 7 11 8 2 4
a b c d e f g h i
4 8 7 9 14 10 2 1 6 7 11 8 2 4
a b c d e f g h i
4 8 7 9 14 10 2 1 6 7 11 8 2 4

An example, Cont'd



Cont’d



a b c d e f g h i
4 8 7 9 2 11 7 6 4 14 10 8 1 2
a b c d e f g h i
4 8 7 9 2 11 7 6 4 14 10 8 1 2
a b c d e f g h i
4 8 7 9 2 11 7 6 4 14 10 8 1 2
4 8 7 9 2 11 7 6 4 14 10 8 1 2

Prim's algorithm: Implementation

We need to keep a list for the vertices not covered by A, and we hope that each time we can efficiently pick the closest one to include. Thus we need a Priority Queue.

Extra variables:

- Q: a min-priority queue to store the vertices which are not in V_A yet.
- **key**: for each element v (a vertex) in Q, there is a field **key** to record the minimum weight of any edge connecting v to a vertex in the tree; i.e., $key[v]$ tells the distance from v to the tree A. If no such edge, $key[v] = \infty$.

Prim's algorithm

MST-PRIM $(G(V,E), w, r)$ /* r is the arbitrarily selected starting point */

for each $u \in V$
    $key[u] := \infty$;

key[r] := 0; /* the first to be picked into V_A */
$Q := V$; /* put all vertices into a PQ */

while Q is not empty
    $u :=$ Extract-Min(Q); /* get the vertex which is closest to the tree A, and remove it from the queue */
    for each $v \in$ Adj[u] /* update the dist. to A */
        if $(v \in Q)$ and $w(u,v) < key[v]$
            key[v] $:= w(u,v)$

Prim's algorithm: Complexity

Use a **Binary Heap** to implement the min-priority queue

```
MST-PRIM (G(V,E), w, r)
for each u ∈ V
    key[u] := ∞;

key[r] := 0;
Q := V;                             — Build-Min-Heap: O(|V|)

while Q is not empty                — Totally execute |V| times
    u := Extract-Min(Q);            — Extract-Min: O(lg|V|)

    for each v ∈ Adj[u]             — What about this part?
        if (v ∈ Q) and w(u,v) < key[v]
            key[v] := w(u,v)
```

Using Binary Heaps

If we use a Heap to implement the min-priority queue:

- Build-Min-Heap (line 4) takes $O(|V|)$.
- **while** loop (line 5) will execute $|V|$ times.
- Extract-Min (line 6) takes $O(lg|V|)$.
- The **for** loop in lines 7 - 9 is executed $O(|E|)$ times altogether, because the sum of the lengths of all adjacency lists is $2|E|$.
- line 8: $O(1)$
- line 9: It's actually an operation of **Decrease-Key**.
  With Heap: $O(lg|V|)$.

Overall the complexity for Prim's algorithm:
$O(|V| + |V|lg|V| + |E|lg|V|) = O(|E|lg|V|)$.