The Selection Problem

- Definition
  - Given an array L containing n keys, find the ith smallest (or largest) key in L ($1 \leq i \leq n$).
- Different cases
  - if $i = 1$, find the smallest key
  - if $i = 2$, find the second smallest key
  - by **median**, we mean:

$$i = \begin{cases} (n+1)/2 & \text{if } n \text{ is odd} \\ \lfloor (n+1)/2 \rfloor & \text{if } n \text{ is even} \end{cases}$$

    (**tell the difference between median and average**).
  - if $i = n$, find the largest key

First Try: Sorting

- The solution is trivial:
  1. Sort the sequence.
  2. Choose the ith element from the sorted sequence.
- What is the complexity?

Can we do better than this?

Problem 1: Finding the smallest key

MINNUM(A)

```
min:= A[1];

for i:=2 to n do
    if (min > A[i])
        min := A[i];

return min;
```

Complexity: $n - 1$ comparisons (Note: this is the exact running time, not an asymptotic one)

Problem 2: Find the minimum and maximum simultaneously (straightforward way)

FIND-BOTH(A)

```
min := A[1];
max := A[1];

for i:=2 to n do
    if (min > A[i])
        min := A[i];
    if (max < A[i])
        max := A[i];

return min, max;
```

Complexity: $2(n-1)$ comparisons (same as finding the largest and smallest keys independently)

Can we do better?

A smarter way:

- Pair the keys and find the minimum and maximum in each pair (about $n/2$ comparisons)
- Collect the smaller keys in a list and find the smallest (about $n/2$ comparisons)
- Collect the larger keys in a list and find the largest (about $n/2$ comparisons)
- Total number of comparisons:

Algorithm

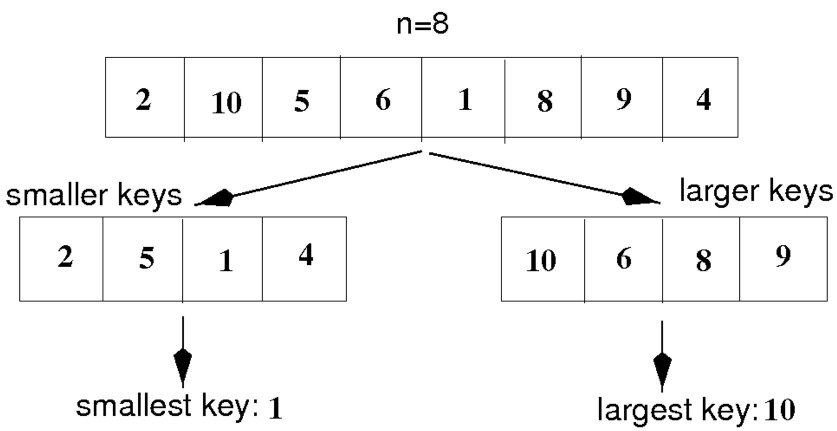
FIND-BOTH-SMARTER(A, n)

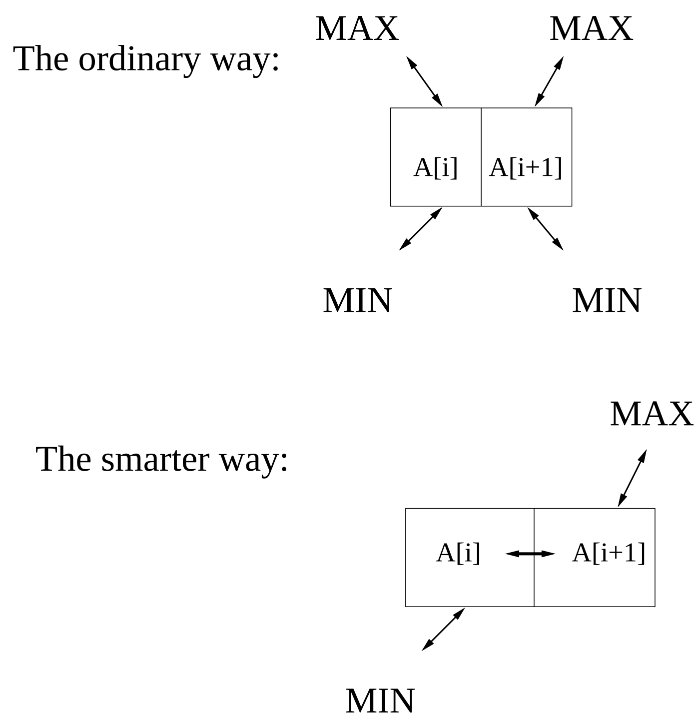
```
if n is odd
    k := 2;
    min := A[1];   max := A[1];
else
    k := 3
    if A[1] < A[2]
       min := A[1];   max := A[2];
    else
       min := A[2];   max := A[1];

for i := k to n − 1 by 2 do
    if A[i] > A[i + 1]
          exchange A[i] and A[i + 1];
    if A[i] < min
          min := A[i];
    if A[i + 1] > max
          max := A[i + 1];
```

What makes the difference here?

Using the ordinary way, each pair require 4 comparisons. With the "smarter" way, the number of comparisons is reduced to 3.





Problem 3: Find the ith smallest key

Idea: Divide and Conquer

Divide: split the input array recursively (using the routine "Partition" (in QuickSort))

Conquer: recursively solve **ONE** sub-problem (Process only the subarray which contains the ith smallest key (note that QuickSort processes both subarrays!))

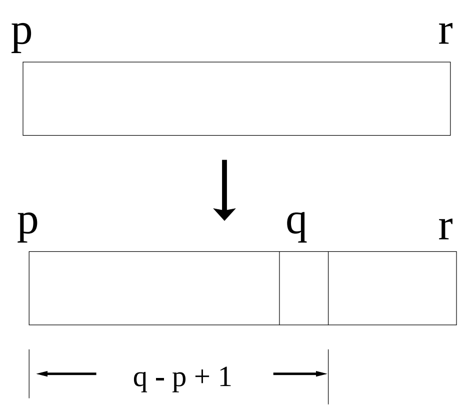
Combine: no need to combine

Algorithm: first try

SELECT(A, p, r, i) /*Find the ith smallest element in A[p..r] */

```
if (p == r) return;

q := Partition(A, p , r);
```




```
k := q - p + 1;

if (i == k)
  return A[q];
else if (i < k)
        return Select(A, p , q-1, i);
    else
        return Select(A, q + 1, r, i-k);
```

Complexity for the first try

- If the partition is balanced (q = n/2), we have $T(n) =$?
- Worst Case, when Partition always results in 2 subarrays with 0 and $n-1$ elements: $T_w(n) =$?

  When will the worst-case happen?

Second Try: Selection in Worst-Case linear time

Basic Idea: to find a split element q such that we always eliminate a fraction α of the elements:

$$T(n) \leq T((1-\alpha)n) + \Theta(n) \text{ then } T(n) = O(n)$$

- For example, each time, if we can guarantee to eliminate at least 10% elements, then $T(n) \leq T(0.9n) + cn$.

  Since $T'(n) = T'(0.9n) + cn \Rightarrow T'(n) = \Theta(n)$,

  Then $T(n) \leq T(0.9n) + cn \Rightarrow T(n) = O(n)$.

Selection with Linear Time in Worst-Case

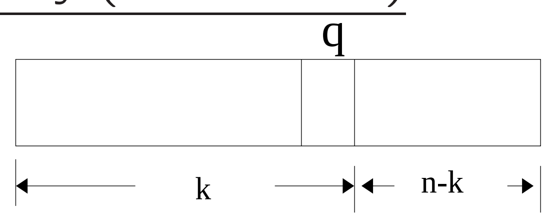
SELECT(i)

1 Divide n elements into groups of 5.

2 Select median of each group ($\Rightarrow \lceil \frac{n}{5} \rceil$ selected elements)

3 Use SELECT recursively to find median q of the medians

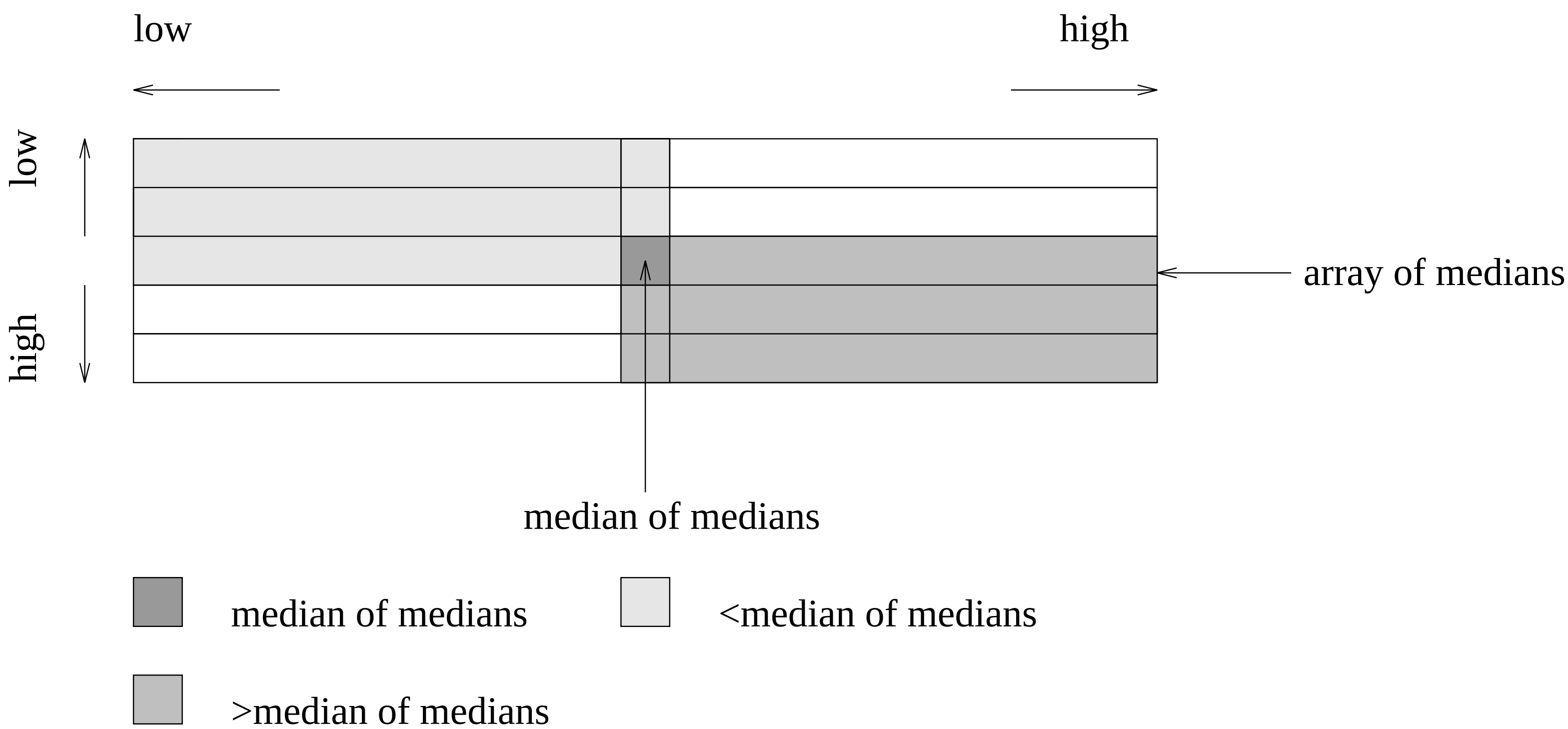
4 Partition the array (all elements) based on q



5 Use SELECT recursively to find ith element

- if $i == k$, we are done
- if $i < k$, then SELECT(i) on $k-1$ elements
- if $i > k$, then SELECT(i - k) on $n-k$ elements

How the algorithm works



Analysis

As our first step in the analysis, we are going to find a lower bound on the # of elements that are greater than the partitioning element s.

- at least $\frac{1}{2}$ of the medians found in step 2 are greater than or equal to s;

- at least $\frac{1}{2}$ of the $\lceil \frac{n}{5} \rceil$ groups contribute 3 elements that are $> s$, except for the one group that has fewer than 5 elements and the one group containing s itself;

- Thus the number of elements $> s$ is at least $3(\lceil \frac{1}{2} \lceil \frac{n}{5} \rceil \rceil - 2) \geq \frac{3}{10} n - 6$; (Note: "3" is from "contribute 3 elements"; "$\lceil \ \rceil$" is from "at least"; "$\frac{n}{5}$" is the total number of groups, "-2" is from "except 2 groups")

Analysis, Cont'd

- Similarly, the number of elements that are $< s$ is at least $\frac{3n}{10} - 6$.
- So no matter which sub-array is picked to continue the search, at least $\frac{3n}{10} - 6$ elements will be eliminated; Equivalently to say, the next call for SELECT will have an input size no bigger than $\frac{7n}{10} + 6$.

Linear Time Selection: An Example

Select (i=7,n=25)

24	12	9	21	2
17	13	4	23	18
1	6	19	16	10
25	22	3	5	7
8	11	14	15	20

Example, cont'd

Step 1:

Break the Array a into $\lceil \frac{n}{5} \rceil = 5$ groups of 5.

Step 2:

Sort each group of 5 elements using the insertion sort. This can be done using 8 comparisons.

2	4	1	3	8
9	13	6	5	11
12	17	10	7	14
21	18	16	22	15
24	23	19	25	20

Example, cont'd

Step 3:

Find the median of median of medians found in step 2. 12 is the median of medians in this case.

Step 4:

Partition the array about the median of medians.
Lower side: 2 9 12 1 6 10 3 5 7 11 4 8
Upper side: 21 24 17 18 23 14 15 20 16 19 22 25 13
So, $k = 12$

Example, cont'd

Step 5:

Call select recursively on

2 9 12 1 6 10 3 5 7 11 4 8

with $i = 7$

As we saw last time, both the low side and high side of the partition have at most $\frac{7n}{10} + 6$ elements.

Complexity

Step 1: Divide elements into groups of 5; $\Theta(n)$

Step 2: To find the median of 5 elements requires constant time; total $\lceil\frac{n}{5}\rceil$ groups, so $\Theta(n)$.

Step 3: Total $\lceil\frac{n}{5}\rceil$ medians; To find the median of medians (a selection problem): $T(\lceil\frac{n}{5}\rceil)$

Step 4: Partition takes linear time: $\Theta(n)$.

Step 5: Recursively call SELECT with input size equal or smaller than $\frac{7n}{10}+6$, complexity for this step: $\leq T(\frac{7n}{10}+6)$.

Overall:

$$T(n) \leq T(\frac{7n}{10}+6) + T(\lceil\frac{n}{5}\rceil) + \Theta(n)$$

Analysis, cont'd

Note:

$\frac{7n}{10} + 6 < n$ for all $n > 20$ and let's take $n \leq 140$ (nothing special about 140, you will see) as small size problems, and it takes constant time to solve them $O(1)$.

We will use the following recurrence relation for $T(n)$:

$$T(n) \leq \begin{cases} \Theta(1) & \text{if } n \leq 140 \\ T(\lceil \frac{n}{5} \rceil) + T(\frac{7n}{10} + 6) + \Theta(n) & \text{if } n > 140 \end{cases}$$

We can show that $T(n) = O(n)$ by substitution.

$T(n) = O(n)$

Proof using the Substitution Method:

Basis:
Assume that $T(n) \leq cn$ for some constant c and all $n \leq 140$. This is true by assumption. (However, we have not specified c, yet).

Induction Step
Assume that $T(n) \leq cn$ holds for all $1 \leq n \leq k-1$, or all numbers in $\{1, 2, ...k-1\}$,

Induction Step

We want to show that $T(n) \leq cn$ also holds for $n = k$, or $T(k) \leq ck$

$$T(k) \leq T(\lceil \frac{k}{5} \rceil) + T(\frac{7k}{10} + 6) + ak$$

$$\leq c\lceil \frac{k}{5} \rceil + c(\frac{7k}{10} + 6) + ak$$

(by Induction Hypothesis, and because $\lceil \frac{k}{5} \rceil$ and $\frac{7k}{10} + 6$ are both in {1, 2, .. k-1 })

$$\leq c(\frac{k}{5} + 1) + c(\frac{7k}{10} + 6) + ak$$

(by the definition of $\lceil\ \rceil$)

$$= 9ck/10 + 7c + ak$$

$$= ck + (-ck/10 + 7c + ak)$$

Cont'd

- We want to prove that: $\exists\ c$, such that $T(k) \leq ck$;

  We can get this done by simply check if it is possible that $(-ck/10 + 7c + ak) \leq 0$.

  When $n > 70$, $(-ck/10 + 7c + ak) \leq 0 \Leftrightarrow c \geq \frac{10ak}{k-70}$,

  so here (assume $n > 140$), we can choose a constant $c \geq 20a$,

  then $T(k) \leq ck$. **End of proof.**

(Note: nothing special with 140; we could replace it by any integer strictly greater than 70 and then choose c accordingly)