HW Assignment 1 (Due date: February 6 by 9:00 am)

1. [**Proof by MI, 30 points**] Use mathematical induction to prove the following statements:

   a. For any set A, $|2^A| = 2^{|A|}$.

   b. For any sequence S with n distinct elements, the total number of permutations of S is $n!$.

   c. The number of edges in a complete directed graph is n^2.

2. [**Tree properties, 20 + 10 points**] Let T be a free tree i.e., a connected acyclic undirected graph. Prove or disprove the following statements:

   a. For any two vertices in T, there is exactly one path between them.

   b. Removing an arbitrary edge from T results in two disjoint trees that between them cover all vertices of T.

   c. (*) The number of edges in T is equal with the number of vertices minus 1.

3. [**Proofs, 10 points**] For an arbitrary set A, let $f : 2^A \rightarrow 2^A$, $f(X) = A - X$. Is f a bijective function? Provide a proof or a counterexample.

4. [**Counting, 10 points**] Let A be a set with m elements and $A_1 \subseteq A$ a subset with n elements, $n \leq m$. How many subsets $A_2 \subseteq A$ are there such that $|A_2 \cap A_1| = 3$? Provide a proof.

5. [**Counting, 10 points (*)**] Let G be a complete undirected graph with n vertices. A *triangle* of G is a cycle of length 3 in G. How many triangles are in G? Provide a proof.

6. [**Running Time, 10 points**] Problem 1-1, columns 1 & 7, page 14.

7. [**Insertion Sort, 10 points**] Describe using pseudocode the insertion sort algorithm implemented with the Romanian folk dance (see class website for the link). Is this algorithm more efficient, the same, or less efficient than the algorithm discussed in class? Explain.

8. [**Correctness, 10 points**] Exercise 2.1-3, page 22.

9. [**Correctness, 10 points**] A thief considers stealing merchandise from a store. The loot is in the form of n items, each with weight w_i pounds and value p_i dolars, for $1 \leq i \leq n$. The thief has a car that can carry a maximum of m pounds. Any item can be put in the car as long as the weight limit m is not exceeded. The problem that the thief needs to solve is: which items should he steal from the store to maximize his profit?

   Let x_i be a binary variable that denotes whether an item is stolen ($x_i = 1$) or left in the store ($x_i = 0$). Below is a greedy algorithm for this problem. Is this algorithm correct? Prove its correctness, or provide a counterexample.

```
void GreedyThief(m, n, p, w)
// p[1..n] and w[1..n] contain the profits and weights respectively of the n objects.
// m is the car capacity and x[1..n] is the solution vector.
    sort the objects such that p[i]/w[i] ≥ p[i+1]/w[i+1].
    // initialize x.
    for i := 1 to n
        x[i] = 0.0;
    // initialize car capacity U.
    U := m;
    // greedily choose which objects to steal.
    for i := 1 to n
        if (w[i] > U)
            continue;
        // put the whole object in the car.
        x[i] := 1.0;
        // decrease car capacity.
        U := U − w[i];
```