7 SOME OTHER PROBLEMS

7.1 Description of main results

In this chapter solutions are given to some problems closely related to the basic scheme, a short review is made of other authors' works related to the theme of this book, and some open and little-studied questions are presented.

In the previous chapters we mainly considered problems of the maximization of number of successes (minimization of loss). Analysis shows that for such problems over a long time interval optimal strategies do not usually coincide with *myopic strategies*, i.e. strategies optimal over short time intervals (in discrete time—for one step).

This can be explained heuristically because for the case of a far horizon, i.e. long time interval, part of the effort is optimally used in the determination of which hypothesis is true. However, discrimination of the true hypothesis is not a direct aim of the solution and is not directly reflected in the form of criterion functional. It would be interesting to find the optimal strategy in a problem in which the discrimination of the true hypothesis is the direct purpose, i.e. where for fixed *a priori* distribution and given observation time (fixed horizon) the probability of accepting a false hypothesis must be minimized. The first steps in this direction are made in §7.2, where the problem of discrimination of the true hypothesis is formulated for 2×2 hypothesis matrices in the discrete time case. It is not difficult to show that in such a problem the solution is to make a decision at the horizon (a terminal control) based only on the value of the *a posteriori* probability at the horizon ν and to decide that for $\xi(\nu) \geq 1/2$ the first hypothesis is true and that for $\xi(\nu) < 1/2$ the second is true. Since the discrimination of hypotheses depends on the values of $\xi(\nu)$, whose distribution in turn depends on the strategy used, the problem reduces to the optimal organization of the observation process. In the problem of loss minimization the optimal strategy always has a threshold character (see Theorem 3.2) and for *all* symmetric matrices (for $\lambda_1^1 < \lambda_1^2$) is the same and is independent of the number of observations remaining. We show that in the symmetric problem of discrimination of hypotheses the optimal strategy depends on the ratio of the absolute values of jumps (in the η coordinates), i.e. on ratio $\rho = |\gamma^1|/\gamma^0$, where $\gamma^1 := \ln(\lambda_1^1/\lambda_1^2)$, $\gamma^0 := \ln[(1-\lambda_1^1)/(1-\lambda_1^2)]$, and it does not always have a threshold character.

We do not have a description of optimal strategies for all possible ρ and restrict ourselves to the study of the cases $\rho := k$ and $\rho := 1/k$, where k is a natural number greater than 1, and we also make some remarks with respect to the case $\rho = 3/2$.

In §7.3 we treat the problem of maximizing the probability of the first jump in a fixed time interval for a hypothesis matrix of size $m \times N$ in continuous time. Since the criterion functional has a simple character, it is possible to complicate the character of probability distribution of jump times. Theorem 7.2 gives necessary and sufficient conditions for the optimality of a strategy in terms of mapping connected with the structure of the hypothesis matrix. The contents of this section were first published in Sonin (1976).

The next section presents the ideas by which the Bayesian approach, the framework in which most of this book is written, is rejected. Most popular amongst non-Bayesian approaches is the *minimax* approach, where the "quality" of a strategy is defined with respect to the minimum value of the functional over all hypothesis values for a fixed strategy. Two-armed bandit problems in such a formulation were considered, in particular, in Vogel (1960*a*,*b*), Fabius & van Zwet (1970) and other works. We mention that in our case (a finite number of hypotheses) the optimal strategies derived from the Bayesian formulation sometimes allow one to define the value function for the minimax formulation as well.

In §7.4 the problem of the maximization of the number of jumps is considered in minimax formulation for the case $m = 2$. Here the content of Fabius & van Zwet (1970) and Vogel (1960*a*,*b*) is briefly presented. We also mention that it follows from the results of this section that the loss function for Bellman's case is of asymptotic order $\sqrt{n}$. This completes the proof of Theorem 3.1 from §3.1.

Section 7.5 briefly presents the interesting work of Rothschild (1974), devoted to the control of prices, which was mentioned in Chapter 1. Here also some comments are given regarding the basic assumptions of this work.

In §7.6 we briefly present work of some other authors related to the content of this book which was not mentioned in previous chapters. We also give a list of some unsolved and little-studied problems.

7.2 Discrimination of hypotheses

The problem of the discrimination of the true hypothesis with $\nu - 1$ observations is a discrete time basic scheme problem with $N = m = 2$ over a finite horizon ν, where the cost function f_n equals zero for all $n < \nu$, at terminal time ν the two possible controls 1 and 2 correspond to the index of the accepted hypothesis (not to the device index) and the cost f_ν has the form

$$f_\nu(\theta, a(\nu)) = \begin{cases} 0 & \text{if } \theta_1 = 1,\ a(\nu) = 1 \quad \text{or } \theta_1 = 0,\ a(\nu) = 2 \\ 1 & \text{if } \theta_1 = 1,\ a(\nu) = 2 \quad \text{or } \theta_1 = 0,\ a(\nu) = 1. \end{cases}$$

Formally, to stay in the framework of the basic scheme, acceptance of the i^{th} hypothesis at the last step can be identified with the choice of the i^{th} control, $i = 1, 2$. Here, similarly to §3.4, since $m = N = 2$, the consideration of vectors $\theta := (\theta_1, \theta_2)$, $\xi := (\xi_1, \xi_2)$, $\beta := (\beta^1, \beta^2)$ is replaced by that of scalars $\theta := \theta_1 = 1 - \theta_2$, $\xi := \xi_1$, $\beta := \beta^1$.

According to what was said above in §2.4 (see also §3.4), such a problem can be presented in the form of a Markov model, where the *a posteriori* probabilities are points of the state space and the parameter θ is absent. Transition probabilities are given by formulae (3.54), (3.55) and by (2.23) the cost function $f_n(\xi, a(n))$ is given by

$$\begin{aligned} f_n(\cdot,\cdot) &:= 0 \quad \text{if } n < \nu, \\ f_\nu(\xi, 1) &:= f^1(\xi) := (1 - \xi), \\ f_\nu(\xi, 2) &:= f^2(\xi) := \xi. \end{aligned}$$

By Theorem 2.4, to find the optimal strategy and value function it is sufficient to obtain the sequence of solutions of the optimality equation

$$F_{n\nu}(\xi) = T_{n+1}F_{n+1,\nu}(\xi) \qquad \text{for } n < \nu \tag{7.1}$$

satisfying the boundary condition $F_{\nu,\nu} = 0$. In the problem of discrimination of hypotheses, when the cost for terminal solution differs from zero, the operator T_ν has the form $T_\nu f(\xi) = \min(f^1(\xi), f^2(\xi))$, and the operators T_n for $n < \nu$ do not depend on n and have the form

$$T_n f := Tf = \min(M^1 f, M^2 f), \quad \text{where (see (3.57))}$$

$$M^j f(\xi) = p^j(\xi) f(\Gamma^{j1}\xi) + (1 - p^j(\xi)) f(\Gamma^{j0}\xi), \quad j = 1, 2. \tag{7.2}$$

Therefore, in spite of the fact that strictly speaking the problem considered is temporally nonhomogeneous, for the solution $F_{n\nu}$ of equation (7.1) we have that $F_{n\nu} = F_{\nu-n}$. Therefore, similarly to the homogeneous case, we can transfer to time remaining notation so that n denotes the length of the remaining time interval (i.e. $(n-1)$ is the number of remaining observations).

Equation (7.1) has the form

$$\begin{aligned} F_1(\xi) &= \min(1 - \xi, \xi), \\ F_{n+1}(\xi) &= \min(M^1 F_n(\xi), M^2 F_n(\xi)), \quad n = 1, 2, \ldots . \end{aligned} \tag{7.3}$$

Here the existence of an optimal strategy and the continuity and convexity of F_n for $0 < \xi < 1$ follow from the results of Chapter 2 (Theorem 2.2). As in §3.4, consider the sequence

$$r_n(\xi) := M^2 F_n(\xi) - M^1 F_n(\xi), \qquad n = 1, 2, \ldots \tag{7.4}$$

and let $r_0(\xi) := 2\xi - 1$.

If s observations remain, $0 \le s \le \nu - 1$ and $\xi(\nu - s - 1)$ is such that $r_s(\xi(\nu - s - 1)) > 0$ (correspondingly $r_s(\cdot) < 0$) then any optimal strategy prescribes the use of the first (second) control at the following step, i.e. for $s > 0$ to observe the corresponding device and for $s = 0$ to accept the corresponding hypothesis. If $r_s(\xi(\nu - s - 1)) = 0$, then any behaviour is optimal.

Repeating the discussion in the proof of formula (3.67) we obtain that $r_n(\xi)$ satisfies the following recurrence relation for $n \ge 1$:

$$r_{n+1}(\xi) = M^1[r_n(\xi)]^+ - M^2[r_n(\xi)]^-, \tag{7.5}$$

where $a^+ := \max(0, a)$, $a := a^+ - a^-$.

Consider the case of symmetric hypotheses, where $\lambda_1^1 := \lambda_2^2 := \lambda^1$, $\lambda_1^2 := \lambda_2^1 := \lambda^2$. Without loss of generality assume $\lambda^1 < \lambda^2$ and let

$$\gamma^1 := \ln(\lambda^1/\lambda^2), \quad \gamma^0 := \ln[(1-\lambda^1)/(1-\lambda^2)], \quad \rho := |\gamma^1|/\gamma^0. \tag{7.6}$$

Then

$$\gamma^{21} = -\gamma^{11} = -\gamma^1 > 0, \qquad \gamma^{10} = -\gamma^{20} = \gamma^0 > 0. \tag{7.7}$$

As in §3.4 it is convenient to make a change in variables $\eta := \tilde{\eta}(\xi) := \ln[\xi/(1-\xi)]$ and let

$$\tilde{F}_n(\eta) := F_n(\tilde{\xi}(\eta)), \qquad \tilde{r}_n(\eta) := r_n(\tilde{\xi}(\eta)), \tag{7.8}$$

where $\tilde{\xi}(\eta)$ is a transformation inverse to $\tilde{\eta}(\xi)$.

The operator M^j transforms to $\widetilde{M}^j$ acting according to the formula (see (3.63))

$$\widetilde{M}^j f(\eta) = \tilde{p}^j(\eta) f(\eta - (-1)^j\gamma^1) + (1 - \tilde{p}^j(\eta)) f(\eta - (-1)^j\gamma^0). \tag{7.9}$$

Consider the case $\rho := k$, where k is integer, $k > 1$ and let $\gamma^0 := \gamma$, $-\gamma^1 := k\gamma$. For fixed k and γ, consider the sequence of sets G_n', $n = 0, 1, \ldots$, on the nonnegative half line $\eta \ge 0$ consisting of the half line $\{\eta > nk\gamma\}$ and intervals of length γ separated from each other by a distance γk, i.e.

$$G_n' := \{\eta \ge nk\gamma\} \cup \left(\bigcup_{s=1}^{[nk/(k+1)]} \left\{ nk - s(k+1) \le \frac{\eta}{\gamma} \le nk - s(k+1) + 1 \right\} \right). \tag{7.10}$$

If the left end point of the unit interval closest to zero coincides with the point γ, then we add to the set G_n' the interval $\{0 \le \eta < \gamma\}$. Thus let

$$\begin{aligned} G_n &:= \begin{cases} G_n' & \text{if } n \ne k \pmod{(k+1)} \\ G_n' \cup \{0 \le \eta < \gamma\} & \text{if } n = k \pmod{(k+1)} \end{cases} \\ G_n^1 &:= \{\eta : \eta > 0,\, \eta \notin G_n\}, \qquad G_n^2 := \{\eta : -\eta \in G_n^1\}. \end{aligned} \tag{7.11}$$

Theorem 7.1 *Any optimal strategy for the symmetric problem of the discrimination of the true hypothesis with* $\gamma = \ln[(1-\lambda^1)/(1-\lambda^2)]$ *and* $\rho = k$ *is of the form*

$$\pi_n(\xi) := \begin{cases} 1 & \text{if } \tilde{\eta}(\xi) \in G_n^1 \\ 0 & \text{if } \tilde{\eta}(\xi) \in G_n^2 \\ \text{arbitrary} & \text{otherwise,} \end{cases} \tag{7.12}$$

where $n = 0, 1, \ldots$ *and* $\pi_n(\xi)$ *denotes the probability with which the first control is used if* n *observations remain and the* a posteriori *probability of the first hypothesis equals* ξ.

Proof. From the above discussion the statement of the theorem derives directly from the following relations, which we will prove by induction:

$$\begin{aligned} \tilde{r}_n(\eta) &> 0 \quad \text{if } \eta \in G_n^1 \\ \tilde{r}_n(\eta) &< 0 \quad \text{if } \eta \in G_n^2 \\ \tilde{r}_n(\eta) &= 0 \quad \text{if } \eta \notin G_n^1 \cup G_n^2. \end{aligned} \tag{7.13}$$

By symmetry $\tilde{F}_n(\eta) = \tilde{F}_n(-\eta)$ which implies that $\tilde{r}_n(\eta) = -\tilde{r}_n(\eta)$. Therefore it is sufficient to show (7.13) for $\eta > 0$.

Simultaneously with (7.13) we prove by induction that $r_n(\xi)$ is linear on each interval whose form in η variables is given by $\gamma s \le \eta \le \gamma(s+1)$, where s is a nonnegative integer. To prove this fact we need the following property of the operator M^j which follows directly from its definition (see also (7.9)):

(a) If the function $f(\xi)$ is linear on each interval whose image in η variables has the form $\gamma s \le \eta \le \gamma(s+1)$ or $\gamma s \le -\eta \le \gamma(s+1)$, then $M^j f(\xi)$ is also linear on these intervals.

We turn now to the inductive proof. For $n = 1$, $F_1(\xi) = \min(\xi, 1-\xi)$. Since $M^j(b\xi + d) = b\xi + d$ (see Lemma 3.1), then from (7.9) it follows that

$$\begin{aligned} M^j F_1(\xi) &= F_1(\xi) = 1 - \xi, && \text{if } \tilde{\eta}(\xi) \ge |\gamma^{2-j}|, \\ M^j F_1(\xi) &< F_1(\xi) && \text{if } \tilde{\eta}(\xi) < |\gamma^{2-j}|. \end{aligned}$$

From this and from (7.4) we obtain that $r_1(\xi) = 0$, if $\tilde{\eta}(\xi) \ge k\gamma$ and $r_1(\xi) > 0$, if $\gamma \le \tilde{\eta}(\xi) < k\gamma$. Using property (a) applied to $f(\xi) = F_1(\xi)$, we obtain from (7.4) that $r_1(\xi)$ is linear on any interval of the type $s\gamma \le \eta \le \gamma(s+1)$. But $r_1(\tilde{\xi}(0)) = 0$ and, as was proved, $r_1(\tilde{\xi}(\gamma)) > 0$ and thus $r_1(\xi) > 0$ for $0 < \tilde{\eta}(\xi) < \gamma$. So for $n = 1$ the induction assumption is proven.

Let the induction assumption hold for all $n \le s$. Then $r_s(\xi)$ does not change sign on the interval $\gamma s < \tilde{\eta}(s) < \gamma(s+1)$, and this means that $r_s^+(\xi)$ or $r_s^-(\xi)$ can be taken as $f(\xi)$ in formulation of property (a). From this and from (7.5) we obtain that $r_{s+1}(\xi)$ is linear on each interval of the type $\gamma s \le \tilde{\eta}(\xi) \le \gamma(s+1)$. Further, $\tilde{r}_s(\eta) = \tilde{r}_s(-\eta) \ge 0$ for $\eta \ge 0$. Hence, by (7.5) and (7.9), we have that

$$\tilde{r}_{s+1}(\eta) = \begin{cases} \tilde{p}^2(\eta)\tilde{r}_s(\eta+\gamma) + (1-\tilde{p}^2(\eta))\tilde{r}_s(\eta - k\gamma) & \text{if } \eta > k\gamma \\ \tilde{p}^2(\eta)\tilde{r}_s(\eta+\gamma) & \text{if } \gamma \le \eta \le k\gamma \\ \tilde{p}^2(\eta)\tilde{r}_s(\eta+\gamma) - \tilde{p}^1(\eta)\tilde{r}_s(\gamma - \eta) & \text{if } 0 \le \eta < \gamma. \end{cases} \tag{7.14}$$

First consider the case $\eta > k\gamma$. If $\eta \in G_{s+1}^1$ then from the structure of the sets G_n^1 it follows that either $\eta - k\gamma \in G_s^1$ or $\eta - k\gamma \notin G_s^1$, but then $\eta - k\gamma \in (0,1)$, $s = k \pmod{(k+1)}$, and it follows that $\eta + \gamma \in G_s^1$. In both cases, from (7.14) and the induction assumption for $n = s$, we obtain that $\tilde{r}_{s+1}(\eta) > 0$.

Now let $\eta > k\gamma$ and $\eta \notin G_{s+1}^1$. Then from the structure of the sets G_n^1 it follows that $\eta + \gamma \notin G_s^1$, $\eta - k\gamma \notin G_s^1$ and thus by induction and (7.14) we have that $\tilde{r}_{s+1}(\eta) = 0$.

If $\gamma \le \eta \le k\gamma$, then from $\eta \in G_{s+1}^1$ it follows that $\eta + \gamma \in G_s^1$ and from $\eta \notin G_{s+1}^1$, it follows that $\eta + \gamma \notin G_s^1$. In both cases, from (7.14) and the induction assumption for $n = s$, we obtain that (7.13) holds in this region for $n = s + 1$.

As was proved above, the function $r_{s+1}(\xi)$ is linear on the interval $0 \le \eta(\xi) \le \gamma$ and equals zero for $\tilde{\eta}(\xi) = 0$. Therefore by the structure of set G_{s+1}^1 (7.13) holds on this interval from the relation just proved for $\eta = \gamma$. So Theorem 7.1 is proved. ■

Remark 7.1 If $\pi_s(\xi)$ gives an optimal strategy in the symmetric problem of discrimination of hypotheses with some γ^0 and $\gamma^1 = -\rho\gamma^0$, then the strategy $1 - \pi_s(\xi)$ will be optimal in the problem with $\tilde{\gamma}^0 := -\gamma^1$,

$\tilde{\gamma}^1 := -\gamma^0$, i.e. $\tilde{\rho} = 1/\rho$. Indeed, if we consider the matrix $\{1 - \lambda_i^j\}$ instead of the matrix $\{\lambda_i^j\}$, then an optimal strategy does not change, since this transformation simply corresponds to exchanging the terms "success" and "failure". On the other hand, if we now exchange the device indices, then an optimal action rule *will* change to the opposite action and the hypothesis matrix will have the form $\tilde{\lambda}^1 := 1 - \lambda^2$, $\tilde{\lambda}^2 := 1 - \lambda^1$. ■

Remark 7.2 From the previous remark and Theorem 7.1, it follows that if in symmetric case $\rho = 1/k$, where k is a natural number not equal to 1, then an optimal strategy in the problem of minimization of loss is optimal also in the problem of discrimination of hypotheses. At the same time, for $\rho = k$ the optimal strategies in these problems take opposite actions. ■

Remark 7.3 It can be shown that for the continuous time problem of discrimination of hypotheses an optimal synthesis has the form

$$\alpha^*(s,\xi) := \begin{cases} 0 & \text{if } \xi < 1/2 \\ 1 & \text{if } \xi > 1/2 \\ \neq 1/2 & \text{if } \xi = 1/2, \end{cases}$$

i.e. a solution has the opposite character to that for the problem of maximization of the number of successes. This agrees with the solution for $\rho = k$, since in discrete time $\rho > 1$ if and only if $\lambda_1^1 + \lambda_1^2 < 1$ and continuous time can be considered as the limiting case of discrete time with an unbounded increase of number n of observations with success probabilities λ_i^j/n in each test. ■

As in the minimization of loss problem, in the problem of discrimination of hypotheses for $\rho = k$ and $\rho = 1/k$ there exists an optimal strategy which has a threshold character, i.e. for $\xi < 1/2$ it prescribes the use of one device and for $\xi > 1/2$ the other. We show that with other values of ρ this can fail.

Example 7.1 Consider the symmetric problem of the discrimination of hypotheses with $\gamma^1 = -3\gamma$, $\gamma^0 = 2\gamma$, i.e. $\rho = 3/2$. Property (a) holds in this case too, therefore, as for the case $\rho = k$, we obtain that $r_1(\xi) := M^2 F_1(\xi) - M^1 F_1(\xi)$ is linear and positive on the intervals $(1/2, \tilde{\xi}(2\gamma))$, $(\tilde{\xi}(2\gamma), \tilde{\xi}(3\gamma))$ and otherwise is equal to zero for $\xi \geq 1/2$. So, for $n = 1$ the optimal synthesis has the same character as for the case $\rho = k$. We show that for $n = 2$ the character of the optimal synthesis changes.

Consider $r_2(\xi) := M^2 r_1^+(\xi) - M^1 r_1^-(\xi)$. From the definition of M^j it follows that $M^2 r_1^+(\xi)$ is piecewise linear and positive on the intervals $(\tilde{\xi}(-2\gamma), \tilde{\xi}(\gamma))$ and $(\tilde{\xi}(3\gamma), \tilde{\xi}(6\gamma))$ and by symmetry the function $M^1 r_1^-(\xi)$ is positive and piecewise linear on $(\tilde{\xi}(-\gamma), \tilde{\xi}(2\gamma))$. From this it follows that $r_2(\xi)$ is strictly negative on $(\tilde{\xi}(\gamma), \tilde{\xi}(2\gamma))$ and, by linearity and equality to zero for $\xi = 1/2$, is also strictly negative on $(\tilde{\xi}(0), \tilde{\xi}(\gamma))$. So, for $\xi \geq 1/2$ the interval $(1/2, \tilde{\xi}(2\gamma))$ on which the second control must be used is exchanged for the interval $(\tilde{\xi}(2\gamma), \tilde{\xi}(3\gamma))$, where any control is optimal, and subsequently by the interval $(\tilde{\xi}(3\gamma), \tilde{\xi}(6\gamma))$, where the first control must be used. ■

7.3 Maximization of first jump probability

In this section we consider problems closely related to the basic scheme (in continuous time) for profit functions q_i of the type $q_1 := 1$, $q_i(t,x) = 0$ for $i \geq 1$. In other words, consider the problem of maximization of the probability of at least one jump (a realization of a 1) in a fixed time interval ν. In this case the strategies β do not depend on the observations, i.e. β coincides with control up to the first jump $\alpha_0(s|\xi) := \alpha$. Therefore in this section we will speak about *controls* instead of strategies (or action rules).

However, now we do *not* assume that the distribution of the first jump time with respect to the j^{th} coordinate is exponential, i.e. the observation process is *not* a Poisson process. Instead of the density function $\exp(-\lambda_i^j \int_0^s \alpha^j(v)\,dv)$ we will consider arbitrary nonincreasing logarithmic convex functions of the variable $\int_0^s \alpha^j(v)\,dv$.

We show that such a problem is equivalent to a convex programming problem of optimization theory. For this problem, we give a description of the set of optimal controls for each initial point ξ, and also prove the existence and continuity with respect to ξ of the derivatives of the value functions $F_\nu(\xi)$. As was mentioned in §1.9 and in

§6.2, this fact is of interest in the consideration of similar problems in which the number of observations is greater than one.

Let $\mathcal{A}_\nu$ be the set of measurable vector-valued functions taking values in S^m and defined on the interval $[0,\nu]$ and let $R_i^j(q)$, $i = 1,\dots,N$, $j = 1,\dots,m$ be nonincreasing continuously differentiable functions of a scalar variable q such that $r_i^j(q) := -\ln R_i^j(q)$ is a convex function, $R_i^j(q) \le 1$ and for each j we may find i such that $R_i^j(q)$ is strictly decreasing.

We assume that the probability of no jump in the j^{th} coordinate under the i^{th} hypothesis on the time interval $[0,s]$ using the control $\alpha(\cdot) \in \mathcal{A}_\nu$ is given by $R_i^j(\int_0^s \alpha^j(v)\,dv)$. Since we are interested in the maximization of the probability of at least one jump with *a priori* distribution of the probability of the first hypothesis ξ, then the functional to be considered is

$$F_\nu^\alpha(\xi) = 1 - \sum_{i=1}^N \xi_i \prod_{j=1}^m R_i^j\left(\int_0^\nu \alpha^j(v)\,dv\right). \tag{7.15}$$

Therefore the value $F_\nu^\alpha(\xi)$ is the same for all $\alpha \in \mathcal{A}_\nu$ such that $\int_0^\nu \alpha^j(s)ds = q^j$, where $q := (q^1,\dots,q^m) \in Q_\nu$ and

$$Q_\nu := \{q = (q^1,\dots,q^m):\ \ q^j \ge 0,\ j = 1,\dots,m,\ \sum_{j=1}^m q^j = \nu\}. \tag{7.16}$$

Using the equality $R_i^j(q) := \exp\{-r_i^j(q)\}$ and denoting

$$-\ln \prod_{j=1}^m R_i^j(q^j) := \sum_{j=1}^m r_i^j(q^j) := R_i(q), \tag{7.17}$$

the initial problem can be rewritten as the following equivalent convex programming problem.

Given $\nu > 0$, $\xi \in S^N$, minimize

$$\sum_{i=1}^N \xi_i \exp\{-\sum_{j=1}^m r_i^j(q^j)\} = \sum_{i=1}^N \xi_i \exp\{-R_i(q)\} \tag{7.18}$$

with respect to all possible $q = (q^1,\dots,q^m) \in Q_\nu$.

We also call the elements of set Q_ν *controls*. Define

$$p^j(\xi,q) := \sum_{i=1}^m \xi_i \frac{dr_i^j}{dq^j}(q^j), \quad \xi \in S^m,$$

$$I(\xi,q) := \{j : p^j(\xi,q^j) = \max_{1\le k\le m} p^k(\xi,q^k)\} \tag{7.19}$$

and call a pair (ξ,q) *admissible* if $j \notin I(\xi,q)$ implies that $q^j = 0$.

We write the values of the *a posteriori* probabilities of hypotheses ξ' at time ν for initial point ξ using the control $\alpha(\cdot) \in \mathcal{A}_\nu$ in the event of no jump up to time ν. By Bayes' formula we have

$$\xi_i' = \xi_i \exp\{-R_i(q)\} / \sum_{k=1}^N \xi_k \exp\{-R_k(q)\} \qquad i = 1,\dots,N, \tag{7.20}$$

where $q^j := \int_0^\nu \alpha^j(s)\,ds$, $j = 1,\dots,m$.

Denote by $\Phi_{\nu,q}$ the map $S^m \to S^m$ given by formula (7.20) for fixed ν, $q \in Q_\nu$.

It is easy to check that the inverse map $\Phi_{\nu,q}^{-1}$ (for fixed ν, q) is given by the formula

$$\xi_i = \xi_i' \exp\{R_i(q)\} / \sum_k \xi_k' \exp\{R_k(q)\}. \tag{7.21}$$

The normalizing multiplier in (7.21) we denote by

$$K(\xi',q) := \{1/\sum \xi_k' \exp\{R_k(q)\}. \tag{7.22}$$

From the continuity of the functions $r_i^j(q^j)$ and (7.22) it can be concluded that for all $q \in Q_\nu$, where $0 \le \nu \le \nu_0$ and for some c, C

$$0 \le R_i(q) \le C, \qquad\qquad 0 < c \le K(\xi',q) \le C. \tag{7.23}$$

The existence of an optimal control for fixed ξ and ν (or, as we will say, of a (ν,ξ)-*optimal* control) follows directly from the continuity (with respect to q) of the criterion functional (7.18) and the compactness of Q_ν. The description of the set of optimal controls for different ν, ξ can be given in terms of the notion of an admissible pair and the map Φ^{-1}.

Theorem 7.2

(a) *For the control $\bar{q}$ to be (ν, ξ)-optimal it is necessary and sufficient that the pair $(\Phi_{\nu,\bar{q}}\xi, \bar{q})$ be admissible.*

(b) *If $\bar{q}$ and $\tilde{q}$ are (ν, ξ)-optimal controls, then*

$$\Phi_{\nu,\bar{q}}\xi = \Phi_{\nu,\tilde{q}}\xi.$$

(c) *The derivative with respect to ξ of the value function $F_\nu(\xi)$ is continuous with respect to variables ξ, ν for ξ belonging to the interior of simplex S^N.*

In principle the theorem makes it possible to find optimal controls. One needs to find all admissible pairs $\{(\xi', q) : \xi' \in S^m, q \in Q_\nu\}$ and consider the map $\Phi^{-1}_{\nu,q}$ acting on them. Since for each point (ν, ξ) at least one optimal control $\bar{q}$ exists and the pair $(\Phi_{\nu,\bar{q}}\xi, \bar{q})$ will be admissible, each point ξ will have at least one pre-image. From statement (b) of the theorem it follows that for all such pairs ξ' will be the same.

Remark 7.4. The functions $p^j(\xi, q)$ involved in the definition of admissible pairs (see 7.19), like the map Φ, have obvious probabilistic meanings: $p^j(\xi', q)$ is the probability density of a transition of the j^{th} device from the state 0 to the state 1 in a small time interval $[\nu - d\nu, \nu)$ under the condition that the value of the *a posteriori* probabilities of hypotheses at time $\nu - d\nu$ coincides with ξ' and the total resources assigned to devices up to time $\nu - d\nu$ coincide with q, i.e. $\int_0^{\nu - d\nu} \alpha^j(s)\,ds = q^j$, $j = 1, \ldots, m$, and $\alpha^j(s) \equiv 1$ for $s \in [\nu - d\nu, \nu)$. Accordingly, statement (a) of the theorem can be clarified as follows. If at time $\nu - d\nu$ a device can be found with index j such that its investment up to moment $\nu - d\nu$ differs from zero and on the interval $[\nu - d\nu, \nu)$ investment in it is unprofitable (i.e. we can find a device with index j' such that with investment of all resources in it on the interval $[\nu - d\nu, \nu)$ the probability of a jump on this device is greater than that on the device with index j with full investment in it) then the control is nonoptimal on the interval $(0, \nu - d\nu)$. ■

Remark 7.5 In the Poisson case, i.e. for $R_i^j(q) = \exp\{-\lambda_i^j q\}$, the functions $p^j(\xi, q)$ have the form $\sum_i \xi_i \lambda_i^j$ and therefore do not depend on q. The possibility in the Poisson case of constructing an optimal

control in the form of a synthesis on the space t, ξ, where t is the time remaining (see, for example, §6.5), is connected with this fact. ■

Proof. For the characterization of optimal controls we use the Kuhn–Tucker theorem in differential form (see Pshenichni 1969, Theorem 2.6). According to this theorem, for $\bar{q} = (\bar{q}^1, \ldots, \bar{q}^m)$ to be the solution of the minimization problem (7.18) and (7.16) it is necessary that there exist constants a, b^0, $b^1, \ldots, b^m$, not all zero, such that the following relations hold:

$$\begin{aligned} &a \sum_{i=1}^N \xi_i \exp\{-R_i(\bar{q}\} \left(-\frac{dr_i^j}{d\bar{q}^j}(\bar{q}^j)\right) + b^0 - b^j = 0, \\ &b^j \bar{q}^j = 0, \quad b^j \geq 0, \quad j = 1, \ldots, m, \quad a \geq 0. \end{aligned} \tag{7.24}$$

If $a > 0$, then these relations are sufficient. (We mention that the inequality in Theorem 2.6 of Pshenichni (1969) is replaced by equality, since the set X coincides in the case considered with the whole space.)

We show that if $\bar{q}$ is a solution of problem (7.16), (7.18) (which always exists by the existence of a (ν, ξ)-optimal control noted above), then in equalities (7.24) $a > 0$, $b^0 > 0$. Indeed, suppose $a = 0$. For at least one r, $\bar{q}^r \neq 0$ and therefore $b^r = 0$. From (7.24) for $j = r$ it follows that $b^0 = 0$ and $b^j = 0$ for all $j = 1, \ldots, m$. This is a contradiction. Similarly, from the condition $dr/dq > 0$ we obtain that $b^0 > 0$.

Dividing (7.24) by $a \sum \xi_k \exp\{-R_k(\bar{q})\}$, calling the definition of the map $\Phi_{\nu,q}$ and denoting $\xi' := \Phi_{\nu,\bar{q}}\xi$ and $b^{j'} := b^j / a \sum \xi_k \exp\{-R_k \bar{q})\}$, $j = 0, 1, \ldots, m$, we rewrite the relations (7.24) as

$$p^j(\xi', \bar{q}^j) = b^{0'} - b^{j'}, \quad j = 1, \ldots, m, \quad b^{j'} \bar{q}^j = 0. \tag{7.25}$$

Now it is easy to see that the pair $(\xi', \bar{q})$ is admissible. Indeed, since $b^{j'} \geq 0$ and for at least one $r > 0$ we have $b^{r'} = 0$, then $\max_k p^k(\xi', \bar{q}) = b^{0'}$. If $p^j(\xi', \bar{q}) < b^{0'}$, i.e. $j \notin I(\xi', \bar{q})$, then $b^{j'} > 0$ and therefore $\bar{q}^j = 0$.

If some pair $(\xi', \bar{q})$, $\bar{q} \in Q_\nu$, is admissible, then for this pair relations (7.25) hold (it is sufficient to set $b^{0'} := \max_k p^k(\xi', \bar{q}^k)$, $b^{j'} := b^{0'} - p^j(\xi', \bar{q}^j)$. By introducing $\xi := \Phi^{-1}_{\nu,q}\bar{q}(\xi')$ and multiplying (7.25) by the required multiplier we obtain that the equations of (7.24) hold and therefore by the second part of the Kuhn–Tucker theorem $\bar{q}$ is a (ν, ξ)-optimal control. So, statement (a) of the theorem is proved.