and let

$$
\begin{aligned}
A_0^1 &:= \{(t,\eta) : t \geq 0,\ \eta < \eta_1^1(t)\} \\
A_0^2 &:= \{(t,\eta) : t \geq 0,\ \eta > l(t) \text{ if } t < t_*,\ \eta > \eta_1^2(t) \text{ if } t > t_*\} \\
\widehat{A}_n^j &:= \{(t,\eta) : (-1)^j \eta_{n+1}^j(t) < (-1)^j \eta \leq (-1)^j \eta_n^j(t)\} \\
A_n^j &:= A^j \cap \widehat{A}_n^j \qquad \text{for} \quad n \geq 1,\ j = 1,2.
\end{aligned}
\tag{5.50}
$$

For $(t,\eta) \in A_n^j$, $n = 0,1,2,\ldots$, $j = 1,2$, let $U(t,\eta) := U_n^j(t,\eta)$, where

$$
\begin{aligned}
U_n^j(t,\eta) := t\tilde{p}^j(\eta) + e^{-\lambda_2^j t}(e^\eta + c)^{-1} \sum_{r=1}^{n} f_r^j(\eta - \eta_n^j(t))/(n-r)! \\
\times \{\lambda_2^j \left[\eta + (n-r)\gamma^j\right] / \delta^j\}^{n-r}
\end{aligned}
\tag{5.51}
$$

and the smooth functions $f_r^j(s)$ are defined on $0 \leq s \leq |\gamma^j|$ such that

$$
\begin{aligned}
f_1^j(0) = \frac{d}{ds} f_1^j(0) = 0, \quad f_r^j(0) = f_{r-1}^j(|\gamma^j|), \\
\frac{d}{ds} f_r^j(0) = \frac{d}{ds} f_{r-1}^j(|\gamma^j|) \qquad \text{for } r > 1.
\end{aligned}
\tag{5.52}
$$

By direct substitution it can be seen that $U(t,\eta)$ defined in this way satisfies equation (5.48) on each of the regions A^j, $j = 1,2$ and is continuously differentiable on them. To see this it is sufficient, for example, for $(t,\eta) \in A^j$ to write $U(t,\eta)$ in the form

$$
U(t,\eta) := t\tilde{p}^j(\eta) + C^j(\eta, \eta - \delta^j t) \exp\{-\lambda_2^j t\}/(c + \exp \eta)
$$

and to solve recursively the equation obtained for $C^j(\eta, v)$ which has the form

$$
\frac{\partial C^j(\eta, v)}{\partial \eta} = \frac{\lambda_2^j}{\delta^j}\, C^j(\eta + \gamma^j,\, v + \gamma^j).
$$

We show below that for $t < t_*$, from the condition of continuous differentiability, and for $t \geq t_*$, from the condition of twice continuous differentiability on the curve $l(t)$ the function $U(t,\eta)$, the curve $l(t)$

and the functions $f_r^j(s)$ are uniquely defined sequentially on the regions A_n^j $(n = 1,2,\ldots)$. The curve $l(t)$ and functions $f_r^j(s)$ obtained in this way are infinitely differentiable everywhere except, possibly, at a finite number of points where the second derivatives have a discontinuity of the first type. Further, it will be proved in lemmas that $U(t,\eta)$ constructed in this way also satisfies the relations (5.49), which completes the proof of the theorem.

We mention that for the continuity of $(\partial/\partial t)U(t,\eta)$ on the curve $l(t)$ it is necessary by (5.46) that the condition

$$
L(t, l(t)) \equiv 0 \tag{5.53}
$$

holds. Let

$$
K(t,\eta) := \tilde{p}^1(\eta)U(t,\eta+\gamma^1) - t\tilde{p}^2(\eta+\gamma^1)\tilde{p}^1(\eta) - \tilde{p}^2(\eta) + \tilde{p}^1(\eta)\,. \tag{5.54}
$$

By construction $U(t,\eta) := t\tilde{p}^2(\eta)$ for $(t,\eta) \in A_0^2$. From this, by direct substitution, we have after elementary calculations (see also (5.64))

$$
L(t,\eta) = K(t,\eta) \quad \text{for } t < t_*, \quad l(t) < \eta\,. \tag{5.55}
$$

The construction outlined above will be conducted in two steps. At the first step, we shall show that, from condition (5.53), equality (5.55) and the continuity of $U(t,\eta)$, we may define the point t_*, the curve $l(t)$ for $t \leq t_*$ and the functions $f_r^1(s)$ for $r \leq n_*$, where n_* is a number such that $(t_*, l(t_*)) \in \widehat{A}_{n*}^1$. Here $f_{n*}^1(s)$ is defined only for $0 \leq s \leq l(t_*) - \eta_{n*}^1(t_*)$ and all $f_r^j(s)$, $r \leq n_*$, are infinitely differentiable and satisfy (5.52).

Indeed, if $(t,\eta) \in \widehat{A}_1^1$, then $(t, \eta + \gamma^1) \in A_0^1$, and this means that $U(\eta+\gamma^1, t) = t\tilde{p}^1(\eta+\gamma^1)$, and after elementary transformations (5.53), taking account of (5.55) and (5.54), yields

$$
l(t) = \ln \frac{1 + t\lambda_2^1}{1 + t\lambda_1^1}\,, \tag{5.56}
$$

so that $l'(0) = -\delta^1 \geq \delta^2$.

If $\delta^1 + \delta^2 = 0$, then set $t_* := 0$ and the construction will consist only of the second step. If $\delta^1 + \delta^2 < 0$, then two cases are possible: either at the intersection of the curve (5.56) with the region $\widehat{A}_1^1$ there exists a point t_1 for which $l'(t_1) = \delta^2$ and then we take $t_* := t_1$ to complete the first step in construction of the curve $l(t)$, or at all points of intersection of the curve (5.56) with $\widehat{A}_1^1$ we have $l'(t) > \delta^2$ and then we define t_1 as the root of equation $l(t_1) = \eta_1^1(t_1)$ and continue the process of the first step of the construction of the curve $\eta = l(t)$ (i.e. we assume $t_* > t_1$). From the condition of continuity of the function $U(t,\eta)$ on the curve $\eta = l(t)$ which is

$$U_1^1(t,l(t)) := t\tilde{p}^1(l(t)) + \frac{\exp\{-\lambda_2^1 l(t)/\delta^1\}}{c + \exp l(t)} f_1^1(l(t) - \eta_1^1(t)) = t\tilde{p}^2(l(t)), \quad (5.57)$$

$f_1^1(s)$ is defined for $0 \le s \le l(t_*) - \eta_1^1(t_*)$ in the first case and for $0 \le s \le |\gamma^1|$ in the second case and, by infinite differentiability of $\tilde{p}^2(\eta)$ and $l(t)$, the function $f_1^1(s)$ will possess the same property. Condition (5.52) for $df_1^1(s)/ds$ is verified directly.

We consider now the case $t_* > t_1$. From $(t,\eta) \in \widehat{A}_2^1$ it follows that $(t,\eta+\gamma^1) \in \widehat{A}_1^1$. But since f_1^1 has already been defined, then $U(t,\eta+\gamma^1)$ is defined for $(t,\eta) \in \widehat{A}_2^1$ and the equation $K(t,\eta) = 0$ gives an infinitely differentiable curve $l(t)$ in the region $\widehat{A}_2^1$ (for this it is sufficient that on this curve for $t < t_2$, where t_2 is defined similarly to t_1, the function $\partial K/\partial\eta$ is not equal to 0, as will be shown below in Lemma 5.2).

From the continuous differentiability of $U(t,\eta)$ in a neighbourhood of the line $\eta_1^1(t)$ the continuous differentiability of the curve $l(t)$ in the neighbourhood of the point t_1 follows. The infinitely differentiable function $f_2^1(s)$ is defined for appropriate s from the condition that $U_2^1(t,l(t)) = t\tilde{p}^2(l(t))$. From the continuous differentiability of $l(t)$ in a neighbourhood of the point t_1, it follows that (5.52) holds for $f_2^1(s)$. If $t_2 = t_*$, then the construction of the first step is complete; if $t_2 < t_*$, then $U(t,\eta+\gamma^1)$ is known now for $(t,\eta) \in \widehat{A}_3^1$ and the condition $K(t,\eta) = 0$ defines the curve $l(t)$ in the region $\widehat{A}_3^1$. As shown in Lemma 5.3, $t_* < \infty$ for $\delta^2 > 0$, which means that by conducting similar constructions recursively at the first step, we can eventually find t_* for which $l'(t) = \delta^2$, and the corresponding $f_j^1(s)$.

Thus, constructing the function $U(t,\eta)$ in the region

$$A_* := \{A_0^2\} \cup \{t,\eta : t \ge 0,\ \eta < \delta^1(t - t_*) + l(t_*)\},$$

the function $U(t,\eta)$ is infitely differentiable on $A_* \cap A^1$ except on the lines η_k^1, $k = 1,\ldots,n_*$, along which the second derivatives could have a discontinuity of the first type. In the region $A_* \cap A^2$ the function $U(t,\eta)$ is equal to $t\tilde{p}^2(\eta)$ and is infinitely differentiable.

By this construction, the partial derivatives of the function $U(t,\eta)$ have limits with respect to a sequence of points tending to the curve $l(t)$ from the regions $A_* \cap A^1$ or $A_* \cap A^2$. We denote these derivatives with indices "$-$" and "$+$" correspondingly. Then from equation (5.48) for $U(t,\eta)$ in the region A^1, and also from (5.55) and the fact that $K(t,l(t)) = 0$ for $0 < t < t_*$ (see also (5.64)), we have that

$$\frac{\partial^-}{\partial t} U(t,l(t)) + \delta^1 \frac{\partial^-}{\partial\eta} U(t,l(t)) = \tilde{p}^2(l(t)) + \delta^1 t \frac{d}{d\eta}\tilde{p}^2(l(t)). \quad (5.58)$$

But since $U(t,l(t)) = t\tilde{p}^2(l(t))$ then it is true that

$$\frac{\partial^-}{\partial t} U(t,l(t)) + l'(t) \frac{\partial^-}{\partial\eta} U(t,l(t)) = \tilde{p}^2(l(t)) + l'(t) t \frac{d}{d\eta}\tilde{p}^2(l(t)). \quad (5.59)$$

From $U(t,\eta) = t\tilde{p}^2(\eta)$ for $(t,\eta) \in A_* \cap A^2$ we obtain that $\partial^+ U(t,l(t))/\partial\eta$ and $\partial^+ U(t,l(t))/\partial t$ also satisfy the corresponding relations (5.58) and (5.59). Since $\delta^1 \ne l'(t)$, then from this it follows that

$$\frac{\partial^-}{\partial t} U(t,l(t)) = \frac{\partial^+}{\partial t} U(t,l(t)),$$

$$\frac{\partial^-}{\partial\eta} U(t,l(t)) = \frac{\partial^+}{\partial\eta} U(t,l(t)),$$

and this means that $U(t,\eta)$ is continuously differentiable on the curve $l(t)$.

* * *

Next consider the second step construction.

Let us say that the piecewise smooth curve $l(t)$ for $0 \le t \le \tilde{t}$ and the function $U(t,\eta)$ in the region

$$\tilde{A} = \bigcup_{j=1}^{2} \left\{(t,\eta) : t \ge 0,\ (-1)^j\eta > (-1)^j\left[\delta^j(t - \tilde{t}) - l(\tilde{t})\right]\right\}$$

satisfies *Condition C*, if in the region A_*, the curve $l(t)$ and the function $U(t,\eta)$ coincide with their counterparts constructed at the first step, $\delta^1 < l'(t) < \delta^2$ for $t_* < t < \tilde{t}$, and if in the region $\tilde{A} \cap A^j$, $j = 1, 2$, the function $U(t,\eta)$ satisfies equation (5.48) and is continuously differentiable except, possibly, on a finite number of lines of the type $\eta = \eta_1^1(t) + \gamma$ where the second derivatives of $U(t,\eta)$ can have a discontinuity of the first type. For $t_* < t < \tilde{t}$ on the curve $\eta = l(t)$ the function $U(t,\eta)$ is twice continuously differentiable.

Given a vector l^j with coordinates $\{1, \delta^j\}$, we denote by $\partial/\partial l^j$ the directional derivative in the corresponding direction, i.e.

$$\frac{\partial}{\partial l^j} f(t,\eta) := \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon, \eta + \delta^j \varepsilon) - f(t,\eta)}{\varepsilon}.$$

If $f(t,\eta)$ is continuously differentiable, then $(\partial/\partial l^j)f = (\partial/\partial t)f + \delta^j(\partial/\partial \eta)f$. The lines parallel to the vector l^j in region A^j are called *trajectories*.

For $0 < t < \tilde{t}$, construct $l(t)$ and $U(t,\eta)$ so as to satisfy Condition C in the corresponding region $\tilde{A}$. Then, using twice differentiability, take the derivative with respect to η in equation (5.48) in the region $A^j \cap \tilde{A}$ and write the result and (5.48) itself in the form

$$\frac{\partial}{\partial l^j} U(t,\eta) = \tilde{p}^j(\eta)\left[1 + U(t,\eta+\gamma^j) - U(t,\eta)\right], \tag{5.60}$$

$$\frac{\partial}{\partial l^j}\frac{\partial}{\partial \eta} U(t,\eta) = \frac{\partial}{\partial \eta}\left\{\tilde{p}^j(\eta)\left[1 + U(t,\eta+\gamma^j) - U(t,\eta)\right]\right\}, \tag{5.61}$$

where the equality (5.61) is true also on the lines where the second derivatives have a discontinuity. Moreover, by continuous differentiability on the curve $l(t)$ we have

$$\frac{d}{dt} U(t, l(t)) = \sum_{j=1}^{2} \alpha^j(t)\tilde{p}^j(l(t))\left[1 + U(t, l(t) + \gamma^j) - U(t, l(t))\right], \tag{5.62}$$

where $\alpha^j(t)$ is defined from the conditions

$$\frac{d}{dt} l(t) = \sum_{j=1}^{2} \alpha^j(t)\delta^j, \qquad \alpha^2(t) := 1 - \alpha^1(t). \tag{5.63}$$

Using (5.60), (5.61) and the directly checked relations

$$\delta^1 \frac{d}{d\eta}\tilde{p}^2(\eta) = \delta^2 \frac{d}{d\eta}\tilde{p}^1(\eta) = \tilde{p}^1(\eta)\left[\tilde{p}^2(\eta+\gamma^1) - \tilde{p}^2(\eta)\right]$$
$$= \tilde{p}^2(\eta)\left[\tilde{p}^1(\eta+\gamma^2) - \tilde{p}^1(\eta)\right], \tag{5.64}$$

we may obtain, after not difficult algebraic transformations, that in the region $A^j \cap \tilde{A}$ we have

$$\frac{\partial}{\partial l^j} L(t,\eta) = -X(t,\eta) - \tilde{p}^j(\eta)L(t,\eta), \tag{5.65}$$

$$X(t,\eta) := \sum_{j=1}^{2}(-1)^j \tilde{p}^j(\eta)\left[\tilde{p}^{3-j}(\eta+\gamma^j) + \frac{\partial}{\partial l^{3-j}}\right]U(t,\eta+\gamma^j)$$
$$= -\sum_{j=1}^{2} \tilde{p}^j(\eta)\alpha^j(t,\eta+\gamma^j)L(t,\eta+\gamma^j). \tag{5.66}$$

Here $\alpha^1(t,\eta) := 1 - \alpha^2(t,\eta) = \alpha^*(t,\eta)$, where $\alpha^*(t,\eta)$ is defined in (5.45), and the last equality in (5.66) is derived from the equations

$$\frac{\partial}{\partial l_j} U(t,\eta) = \tilde{p}^j(\eta)\left[1 + U(t,\eta+\gamma^j) - U(t,\eta)\right] + (-1)^{3-j}\alpha^{3-j}(t,\eta)L(t,\eta)$$

for $j = 1, 2$ and from the optimality of $\alpha^*(t,\eta)$ (see (5.46)).

As mentioned before, from the continuous differentiability of the function $U(t,\eta)$ on the curve $l(t)$ (5.53) follows, i.e. $L(t, l(t)) \equiv 0$, and from its twice differentiability the continuous differentiability of $L(t,\eta)$ follows, from which, according to (5.65) and (5.53), it follows that

$$X(t, l(t)) \equiv 0 \text{ for } t_* \le t \le \tilde{t}. \tag{5.67}$$

Now we will use (5.67), (5.62) and (5.60) for further construction of $U(t,\eta)$.

If $l(t)$ and $U(t,\eta)$ have been constructed in the region $\tilde{A}$ for some $\tilde{t} \ge t_*$ so as to satisfy Condition C, then by the first expression for

$X(t,\eta)$ in (5.66), the function $X(t,\eta)$ has also been defined on the region (see Figure 9)

$$B := \bigcap_{j=1}^{2} \Big\{(t,\eta) : (-1)^j \left[\delta^j(t-\tilde{t}) + l(\tilde{t}) - \gamma^j\right] < (-1)^j\eta < (-1)^j \left[\delta^j(t-\tilde{t}) + l(\tilde{t})\right]\Big\}$$

and is continuously differentiable there.

Suppose $X(\tilde{t}, l(\tilde{t})) = 0$ and $(-1)^j(\partial/\partial l^j)X(\tilde{t}, l(\tilde{t})) > 0$. Then $(\partial X/\partial\eta)\ (\tilde{t}, l(\tilde{t})) > 0$ and by the implicit function theorem the relation (5.67) defines in B a continuous curve $l(t)$ for $\tilde{t} < t < \tilde{\tilde{t}}$ for some $\tilde{\tilde{t}}$, and $l(t)$ is infinitely differentiable except at a finite number of points, where $l'(t)$ may have a discontinuity of the first type and $\delta^1 < l'(t) < \delta^2$. Then (5.62) can be considered as an ordinary differentiable equation in $U(t, l(t))$ for $\tilde{t} < t < \tilde{\tilde{t}}$ and known $U(t, l(t)+\gamma^j)$ and (5.60) as an ordinary differential equation in $U(t,\eta)$ with respect to the corresponding directions with known $U(t,\eta+\gamma^j)$ and with initial data from the solution of (5.62).

We show below that, constructed in this way, $l(t)$ and $U(t,\eta)$ will satisfy Condition C in the corresponding region $\tilde{A}$ and $X(t_*, l(t_*))=0$ (the last expression is demonstrated in the proof of Lemma 5.3). If we are able to show that for $l(t)$ defined by (5.67), $\delta^1 < l'(t) < \delta^2$ always holds, and the constructed $U(t,\eta)$ verifies (5.49), then we have constructed the optimal synthesis and value function. However, for $t > t_*$ the needed inequality for $l'(t)$ has been proved only for the case $\delta^1+\delta^2 = 0$. In Lemma 5.2 it will be shown that $(-1)^j(\partial/\partial l^j)L(t,\eta) < 0$ for $\delta^1 + \delta^2 = 0$. From this, by (5.66), it is not difficult to show that $(-1)^j(\partial/\partial l^j)X(t,\eta) > 0$ from which it follows that $\delta^1 < l'(t) < \delta^2$. From formula (5.68), proved in Lemma 5.2, (5.49) follows.

We show now that if $l(t)$ and $U(t,\eta)$ satisfying Condition C have been constructed in region $\tilde{A}$, then $l(t)$ and $U(t,\eta)$ constructed as above in region $\tilde{A}$ $(\tilde{\tilde{t}} > \tilde{t})$ will also satisfy Condition C, if we assume that $\delta^1 < l'(t) < \delta^2$ for $t < \tilde{\tilde{t}}$.

Since $U(t, l(t))$ satisfies equation (5.62), then the theorem about differential dependence on initial data and parameters can be applied to the equations (5.60), considered as ordinary differential equations with respect to the corresponding directions (where $U(t,\eta+\gamma^j)$ is a known function, and $U(t,\eta)$ is unknown) with initial data $U(t, l(t))$ and parameters $v^j := \eta - \delta^j t$, $j = 1,2$. Thus, $U(t,\eta)$ will be infinitely differentiable except, possibly, on the lines parallel to the direction of differentiation where the first derivatives could be discontinuous. Comparing the limiting values for the directional derivatives near the curve $l(t)$ obtained from equations (5.60) and (5.62), we conclude that the function $U(t,\eta)$ is continuously differentiable on the curve $l(t)$ and that $L(t, l(t)) \equiv 0$. From this it is not difficult to obtain in addition that the first derivatives are continuous on the corresponding lines.

From relation (5.65) on the regions $A^j \cap A$ and the identities $L(t, l(t)) \equiv 0$, $X(t, l(t)) \equiv 0$ the continuous differentiability of $L(t,\eta)$ on the curve $l(t)$ follows, and this implies twice differentiability of the function $U(t,\eta)$. Thus we have that the function $U(t,\eta)$ satisfies Condition C in $\tilde{A}$.

* * *

The construction of the curve $l(t)$ and the function $U(t,\eta)$ for the case $\delta^2 < 0$ includes only the first step (see Lemma 5.3) and is conducted as follows. Suppose that $l(t)$ has been constructed for $t < \tilde{t}$ and in the region

$$\tilde{A} := \bigcap_{j=1}^{2} \{(t,\eta) : \eta < \delta^j(t-\tilde{t}) + l(\tilde{t})\}$$

a smooth function $U(t,\eta)$ has been constructed satisfying (5.48) in the corresponding regions A^j which is infinitely differentiable except on the corresponding lines. Then equation (5.48) with $j = 2$ can be considered as an ordinary differential equation in the direction of the corresponding d/dl^2 and solved in region B (see Figure 10), assuming $U(t,\eta)$ is sought and $U(t,\eta+\gamma^2)$ is known, with initial data $U(0,\eta) = 0$. As a result we obtain the function $\tilde{U}(t,\eta)$ on region B. Substitute $\tilde{U}(t,\eta)$, $U(t,\eta+\gamma^1)$ and $U(t,\eta+\gamma^2)$ for $(t,\eta) \in B$ in the expression for $L(t,\eta)$ and subsequently consider the relation $L(t, l(t)) \equiv 0$ as an equation defining the curve $l(t)$ (as was done similarly for $t < t_*$ for the case $\delta^2 \geq 0$) to obtain the curve $l(t)$ for $\tilde{t} < t < \tilde{\tilde{t}}$ (see Figure 10). Set $U(t,\eta) := \tilde{U}(t,\eta)$ on the region $A^2 \cap B$, and then solve equation (5.48) with $j = 1$ on the strip

$$\{(t,\eta) : \delta^1(t-\tilde{t}) + l(\tilde{t}) < \eta < \delta^2(t-\tilde{\tilde{t}}) + l(\tilde{\tilde{t}})\}.$$

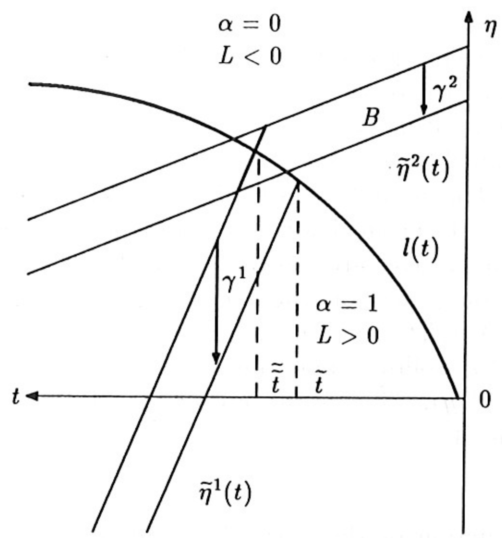

Figure 10
Optimal synthesis for the problem of loss minimization in continuous time with $m = N = 2$ (in coordinates η, t, where t is time remaining) for $\delta^2 < 0$.

which can also be considered as an ordinary differential equation with initial data $\tilde{U}(t, l(t))$. As a result we obtain $U(t,\eta)$ on the region $\tilde{A}$.

* * *

We now formulate Lemmas 5.2 and 5.3, from which the statement of the theorem follows.

If, by the above construction on the strip $H_{\tilde{t}} := \{(t,\eta) : 0 \le t < \tilde{t}\}$, the curve $l(t)$ and the function $U(t,\eta)$ are defined such that either $\delta^2 < l'(t)$ for $t < \tilde{t}$, or there exists $t_* < \tilde{t}$ such that $l(t)$ and $U(t,\eta)$ satisfy Condition C, then we will say that a *synthesis* has been *constructed* on $H_{\tilde{t}}$. Here, $L(t,\eta)$ will be continuously differentiable except on the curve $l(t)$ for $t < t_*$ and in the regions A^j, $j = 1,2$, where there may be some straight lines parallel to the line $\eta = \delta^j t$ where the

derivatives can only have discontinuities of the first type. By (5.66) the corresponding statement is true for $X(t,\eta)$. In inequalities of the type $\partial L/\partial\eta > 0$ we will assume that at points of discontinuity limits from both sides have the appropriate sign.

Lemma 5.2 *Suppose a synthesis has been constructed on the strip* $H_{\tilde{t}} := \{(t,\eta) : 0 \le t \le \tilde{t}\}$. *Then:*

$$(a)\quad \frac{\partial}{\partial\eta} L(t,\eta) < 0 \text{ for all } (t,\eta) \in H_{\tilde{t}},\ \eta \ne l(t) \text{ if } t \ge t_*. \qquad (5.68)$$

(b) For the case $\delta^1 + \delta^2 = 0$,

$$\frac{\partial}{\partial l^1} L(t,\eta) > 0,$$
$$\frac{\partial}{\partial l^2} L(t,\eta) < 0 \text{ for all } (t,\eta) \in H_{\tilde{t}},\ \eta \ne l(t). \qquad (5.69)$$

■

Lemma 5.3 $t_* = \infty$ if $\delta^2 \le 0$, $t_* < \infty$ if $\delta^2 > 0$. ■

The statements of Lemmas 5.2 and 5.3 were used in the construction of the synthesis, and the inequality (5.68) provides the optimality of synthesis on the half plane considered. Indeed, from (5.68) it follows that in the region A^1, $L(t,\eta) > L(t,l(t)) \equiv 0$ holds, and in A^2, the reverse inequality holds.

Before proving Lemma 5.2, we obtain a formula connecting the values $\partial L/\partial\eta$ at different points of a single trajectory.

Suppose $t > t_1$ and that the points (t,η) and (t_1,η_1) lie on the same trajectory in the region A^j $(j = 1,2)$, i.e. $\eta_1 = \eta - \delta^j(t - t_1)$. By (5.65) we have on a single piece of the trajectory (resulting from the piecewise differentiability in η of the right-hand side of (5.65)) the formula

$$\frac{\partial}{\partial l^j}\frac{\partial}{\partial\eta} L(t,\eta) + \tilde{p}^j(\eta)\frac{\partial}{\partial\eta} L(t,\eta) = -N(t,\eta), \qquad (5.70)$$

where (using (5.66))

$$N(t,\eta) := \left[L(t,\eta) - L(t,\eta+\gamma^j)\right] \frac{\partial}{\partial\eta}\,\tilde{p}^j(\eta) - \tilde{p}^j(\eta)\,\frac{\partial}{\partial\eta}\,L(t,\eta+\gamma^j)$$
$$- \alpha^j(t,\eta+\gamma^{3-j})\left[L(t,\eta+\gamma^{3-j})\,\frac{d}{d\eta}\,\tilde{p}^{3-j}(\eta)\right.$$
$$\left. + \tilde{p}^{3-j}(\eta)\,\frac{\partial}{\partial\eta}\,L(t,\eta+\gamma^{3-j})\right]. \quad (5.71)$$

Multiplying (5.70) by the appropriate exponent and integrating along the piece of the trajectory under consideration from t_1 to t, we obtain

$$\frac{\partial}{\partial\eta}\,L(t,\eta) = \exp\{-\int_{t_1}^{t} \tilde{p}^j(\eta-\delta^j(t-u))du\}\,\frac{\partial}{\partial\eta}\,L(t_1,\eta-\delta^j(t-t_1))$$
$$-\int_{t_1}^{t} \exp\{-\int_{s}^{t} \tilde{p}^j(\eta-\delta^j(t-u))du\}N(s,\eta-\delta^j(t-s))\,ds. \quad (5.72)$$

Proof (of Lemma 5.2). First suppose that $\tilde{t} < t_*$. We introduce

$$t_0 := \inf_{s\leq t}\{s := \eta \text{ exists such that } \partial L(s,\eta)/\partial\eta \geq 0\}.$$

It is easily seen that $\partial L(t,\eta)/\partial\eta < 0$ in a neighbourhood of $t = 0$. From this it follows that $t_0 > 0$.

We now consider points $(t_0,\eta) \in A^2$ and apply formula (5.72) for $t := t_0$, $t_1 := 0$. By the definition of t_0, $\partial L(s,\eta)/\partial\eta < 0$ for all $s < t_0$. From this $L(s,\eta+\gamma^2) < L(s,\eta)$ and (for example, for $\delta^2 > 0$), from the inequalities

$$\frac{d}{d\eta}\tilde{p}^1(\eta) < 0,\quad \frac{d}{d\eta}\,\tilde{p}^2(\eta) > 0,\quad \frac{\partial}{\partial\eta}\,L(s,\eta+\gamma^j) < 0,\quad j = 1,2,$$
$$L(s,\eta+\gamma^1) < 0$$

we obtain that $N(s) := N(s,\eta-\delta^j(t_0-s)) > 0$ for all s. (It is easily checked that $N(s) > 0$ also in the case $\delta^2 \leq 0$.) Since $\partial L(0,\eta)/\partial\eta < 0$, then from (5.72) we immediately obtain that $\partial L(t_0,\eta)/\partial\eta < 0$ and this means that

$$\frac{\partial^+}{\partial\eta}\,L(t,l(t)) < 0. \quad (5.73)$$

Notice that, by (5.53) and (5.65),

$$\frac{\partial^\pm}{\partial t}\,L(t,l(t)) + l'(t)\,\frac{\partial^\pm}{\partial\eta}\,L(t,l(t)) \equiv 0$$
$$\frac{\partial^+}{\partial l^2}\,L(t,l(t)) = -X(t,l(t)) \quad (5.74)$$
$$\frac{\partial^-}{\partial l^1}\,L(t,l(t)) = -X(t,l(t)).$$

From this

$$(l'(t)-\delta^2)\,\frac{\partial^+}{\partial\eta}\,L(t,l(t)) = X(t,l(t))$$
$$(l'(t)-\delta^1)\,\frac{\partial^-}{\partial\eta}\,L(t,l(t)) = X(t,l(t)). \quad (5.75)$$

Since $l'(t) > \delta^2 > \delta^1$, then from (5.73) it follows that $X(t,l(t)) < 0$ for $t < \tilde{t}$, and this means that for $t < \tilde{t}$

$$\frac{\partial^-}{\partial\eta}\,L(t,l(t)) < 0. \quad (5.76)$$

Applying a formula similar to (5.72) for $(t_0,\eta) \in A^1$ for the points t_0 and t_1 (where t_1 is the coordinate of the intersection point of the trajectory passing through (t_0,η) with the curve $l(t)$) and using (5.75), we obtain that $t_0 > \tilde{t}$. Thus, statement (a) has been proved for the case $\tilde{t} < t_*$. Notice that in this way it is proved that the construction of the first step described above can be performed at least up to t_*, and the synthesis obtained here is optimal for $t \leq t_*$.

Now we consider the case $\tilde{t} > t_*$. Since (5.74) and (5.75) are true for $t \geq t_*$, then by (5.67) we obtain that

$$\frac{\partial^+}{\partial\eta}\,L(t,l(t)) = \frac{\partial^-}{\partial\eta}\,L(t,l(t)) \quad \text{if } t_* < t < \tilde{t}. \quad (5.77)$$

Putting

$$t_0 := \inf_{s\leq t}\{s : \eta \text{ exists such that } \eta \neq l(s),\ \partial L(s,\eta)/\partial\eta \geq 0\}. \quad (5.78)$$

Similarly to the case $\tilde{t} \le t_*$, we obtain that $\partial L(t_0,\eta)/\partial\eta < 0$ for all $\eta \ne l(t_0)$. Suppose that $t_0 < \tilde{t}$. For $|\eta| > c$, where c is sufficiently large, $L(t_0,\eta)$ can be written in explicit form and it can be checked that (5.68) holds. Therefore, without loss of generality, it can be assumed that the inequality $\partial L(t,\eta)/\partial\eta < 0$ is true for $\{t,\eta : t \le t_0 + a_2, |\eta - \eta(t_0)| > a_1\}$, where a_1 is arbitrary and $a_2 := a_2(a_1)$.

If we show that in some two-dimensional circular neighbourhood of the point $(t_0, l(t_0))$ with radius a_3, $\partial L(t,\eta)/\partial\eta < 0$ except on the turnpike, then we can conclude that $\partial L(t,\eta)/\partial\eta < 0$ for $\{t,\eta : t \le t_0 + a_4, \eta \ne l(t)\}$ for some a_4. So we arrive at a contradiction with the definition of t_0 and our assumption that $t_0 < \tilde{t}$.

We consider a neighbourhood of the point $(t_0, l(t_0))$ of radius a and the parts of the trajectories passing through the points of the interval $\{(t_0,\eta) : l(t_0) - a < \eta \le l(t_0) + a\}$ and through the points of the turnpike $\{(t, l(t)) : t_0 < t \le t_0 + a\}$ (for reasons of clarity only the intersection of the neighbourhood with A^2 is considered). By the above-mentioned inequality for $\partial L/\partial\eta$, for appropriate a_1 and a_2 we have that $\partial L(t,\eta+\gamma^j)/\partial\eta < 0$, $j = 1,2$. Using the continuity of the remaining terms of formula (5.71), and also the inequalities $d\tilde{p}^1/d\eta < 0$ and $d\tilde{p}^2/d\eta > 0$, we obtain $N(t,\eta) > 0$. From this, (5.52) and $\partial L(t,l(t))/\partial\eta = 0$, $\partial L(t_0,\eta)/\partial\eta < 0$ for $\eta \ne l(t)$, we obtain that $\partial L/\partial\eta < 0$ in some neighbourhood of $(t_0, l(t))$ with the exception of turnpike points, which it was required to prove. Thus $t_0 \ge \tilde{t}$ and, moreover, $\partial L(t_0,\eta)/\partial\eta < 0$ for $\eta \ne l(t_0)$. Thus, statement)a) has been completely proved.

The proof of statement (b) of Lemma 5.2 repeats the proof of the negativity $\partial L(t,\eta)/\partial\eta$ given above and is also based on (5.65) and the differential properties of the functions L and X. In the formula analogous to (5.74), $\partial/\partial l^j$ appears instead of $\partial/\partial\eta$ and in some regions (for example, where differentiation with respect to l^j coincides with the direction of the trajectories) a formula similar to (5.70) holds involving second derivatives along the trajectories rather than mixed derivatives. The check of the inequalities (5.69) at $t = 0$ is conducted directly. These inequalities hold only for $\delta^1 + \delta^2 = 0$. The sign of the derivative $\partial/\partial l^1$ is opposite to the signs of $\partial/\partial l^2$ and $\partial L/\partial\eta$, because this condition holds for $t = 0$, and the signs of $d\tilde{p}^1/d\eta$, $d\tilde{p}^2/d\eta$

coincide correspondingly with the signs of $\partial\tilde{p}^1/\partial l^2$, $\partial\tilde{p}^2/\partial l^2$ and are opposite to the signs of $\partial\tilde{p}^1/\partial l^1$, $\partial\tilde{p}^2/\partial l^1$. ∎

Proof (of Lemma 5.3). Consider the case $\delta^2 < 0$ and suppose that $t_* < \infty$. Then a synthesis is constructed on the strip H_{t_*} and, according to (5.75), $X(t_*, l(t_*)) = 0$ necessarily holds. As mentioned above, from Lemma 5.2 applied to the strip H_{t_*} it follows that on this strip $L(t,\eta) > 0$ for all $(t,\eta) \in H_{t_*} \cap A^1$. According to formula (5.66), for the points of the curve $l(t)$ we have

$$X(t, l(t)) = -\tilde{p}^1(l(t))L(t, l(t) + \gamma^1) - \tilde{p}^2(l(t))L(t, l(t) + \gamma^2).$$

Moreover, $\gamma^2 \le 0$ holds for $\delta^2 \le 0$ (see formula (5.4)) and therefore $L(t_*, l(t_*) + \gamma^2) \ge 0$. Since $L(t_*, l(t_*) + \gamma^1) > 0$, we obtain that $X(t_*, l(t_*)) < 0$. Thus the assumption that $t_* < \infty$ is not true.

Consider now the case $\delta^2 > 0$. Assuming that $t_* < \infty$ we construct a curve $\eta = \gamma(t)$ such that $\gamma(t) \ge l(t)$ for all t and $\gamma'(t) \to 0$ as $t \to \infty$, which obviously contradicts the inequality $l'(t) > \delta^2$ (which must hold for all t when $t_* = \infty$).

As mentioned in §4.3, in problems of Bayesian type the value function $V(t,\xi)$ is for fixed t a concave function of ξ and in our case

$$V(t,\xi) < (1-\xi)V(t,0) + \xi V(t,1) = t\left[\lambda_2^1(1-\xi) + \lambda_1^2\xi\right],$$

or, which is the same,

$$U(t,\eta) < t\left[\lambda_1^1 \frac{c}{c+\exp\eta} + \lambda_1^2 \frac{\exp\eta}{c+\exp\eta}\right]. \tag{5.79}$$

We substitute (5.79) in expression (5.54) for $K(t,\eta)$ and using the fact that $K(t,\eta) \equiv L(t,\eta)$ for $\eta > l(t)$ we obtain, for appropriate t and η,

$$L(t,\eta) < \frac{\varepsilon_1}{c+\exp\eta}\left(\exp\eta - 1 - \lambda_2^1 t\right) := R(t,\eta).$$

For $\gamma(t) := \ln(1 + \lambda_2^1 t)$, the function $R(t,\gamma(t))$ equals 0, and since $L(t,\eta) < R(t,\eta)$, then $\gamma(t) > l(t)$. Since $\gamma'(t) \to 0$ as $t \to \infty$ and $l'(t) > \delta^2 > 0$, we come to a contradiction with the assumption that $t_* < \infty$. Lemma 5.3 is proved. ∎

This completes the proof of Theorem 5.3. ∎