we give below only the definitions, similar to those introduced in §3.2 for discrete time, required to formulate the continuous time theorem.

Let $\{\lambda_i^j\}$ be a hypothesis matrix, with $0 \le \lambda_i^j < \infty$. Define

$$\lambda_i := \max_j \lambda_i^j, \qquad R_i := \{j : 1 \le j \le m, \ \lambda_i^j = \lambda_i\}.$$

We say that the hypothesis H_k is a *nuisance* for the hypothesis H_i if $\lambda_k^j = \lambda_i^j = \lambda_i$ for all $j \in R_i$ and $\lambda_i < \lambda_k$. We say that the nuisance H_k is *removable* if there exists a j such that $\lambda_k^j = 0$, $\lambda_i^j \ne 0$. This definition differs little from the definition for discrete time where $0 \le \lambda_i \le 1$ and the values 1 and 0 of λ were playing the same rôle. We call a hypothesis matrix a *B-matrix* if there exists at least one nonremovable nuisance and an *F-matrix* otherwise.

Theorem 5.1

(a) *For F-matrices, $W_\nu(\xi) \nearrow W_\infty(\xi) < \infty$, where $W_\nu(\xi)$ denotes the loss function over the horizon $\nu < \infty$ (see (4.9), (4.10)).*

(b) *For B-matrices, $W_\nu(\xi) \nearrow +\infty, \lim_{\nu \to \infty} W_\nu(\xi)/\nu = 0$.* ∎

Similarly to Corollary 3.1, we have the following.

Corollary 5.1 *For any matrices*

$$\lim_{\nu \to \infty} V_\nu(\xi)/\nu = \sum_{i=1}^N \xi_i \lambda_i.$$
∎

5.2 Minimization of loss over an infinite horizon for the case $m = N = 2$

In the case of two hypotheses and two devices ($m = N = 2$) by virtue of the relations $\xi_1(t) + \xi_2(t) \equiv 1$, $\beta^1(t) + \beta^2(t) \equiv 1$ it is convenient, as in discrete time, to consider the scalars $\xi := \xi_1$ and $\beta := \beta^1$ instead of the vectors $\xi = (\xi_1, \xi_2)$ and $\beta = (\beta^1, \beta^2)$ and, instead of the functions $W_s^\pi(\xi_1, \xi_2)$, $W_s(\xi_1, \xi_2)$ and $\alpha(\xi_1, \xi_2)$, to consider

$$W_s^\pi(\xi) := W_s^\pi(\xi, 1 - \xi), \qquad W_s(\xi) := W_s(\xi, 1 - \xi)$$

and $\alpha(\xi) := \alpha^1(\xi, 1 - \xi)$.

Define, as before,

$$\delta^j := \lambda_1^j - \lambda_2^j, \quad \varepsilon_i := \lambda_i^1 - \lambda_i^2, \quad \varepsilon := \varepsilon_1 - \varepsilon_2 = \delta^1 - \delta^2. \qquad (5.1)$$

Changing the indices of devices and hypotheses if necessary, we will assume in the sequel without loss of generality that

$$|\delta^1| \ge |\delta^2|, \qquad \delta^1 \le 0. \qquad (5.2)$$

Note that if $\varepsilon_1 \cdot \varepsilon_2 < 0$, then taking account of (5.2) it is easily seen that $\delta^1 \ne \delta^2$, $\delta^1 < 0$, $\varepsilon_1 < 0$, $\varepsilon_2 > 0$. Thus, all hypothesis matrices Λ can be divided into the following five groups (see also §3.4):

(O) $\varepsilon_1 \cdot \varepsilon_2 \ge 0$,

(A) $\varepsilon_1 < 0, \ \varepsilon_2 > 0, \ \delta^1 < \delta^2 < 0$,

(B) $\varepsilon_1 < 0, \ \varepsilon_2 > 0, \ \delta^1 < \delta^2 = 0$, $\qquad\qquad$ (5.3)

(C) $\varepsilon_1 < 0, \ \varepsilon_2 > 0, \ 0 < \delta^2 < -\delta^1$,

(D) $\varepsilon_1 < 0, \ \varepsilon_2 > 0, \ 0 < \delta^2 = -\delta^1$.

In Case O, Λ has a column all of whose elements are not less than the elements of the other column; obviously an optimal strategy prescribes the permanent use of the device corresponding to this column. In this case $W(\xi) \equiv 0$. Moreover, in all cases $W(0) = W(1) = 0$, therefore it is assumed below that $0 < \xi < 1$ and Case O is not considered further.

Define (see (5.1))

$$\gamma^j := \ln \frac{\lambda_1^j}{\lambda_2^j}, \quad p := \frac{\varepsilon_1}{\delta^1 - \lambda_1^1 \gamma^1}, \quad q := \frac{\varepsilon_2}{\delta^2 - \lambda_2^2 \gamma^2}, \quad r := -\frac{\varepsilon_2}{\delta^2 - \lambda_1^2 \gamma^2}.$$
$$(5.4)$$

If $\lambda_1^1 = 0$, we will set $p := -\varepsilon_1/\lambda_2^1$ and if $\lambda_2^2 = 0$, then $q := \varepsilon_2/\lambda_1^2$. Using (5.2) it is easy to check that

$$p > 0, \qquad q > 0, \qquad r > 0. \qquad (5.5)$$

Theorem 5.2 *In Case B, $W(\xi) = \infty$. In Cases C and D the strategy given by the synthesis*

$$\alpha^*(\xi) := \begin{cases} 1 & \text{if } \overset{\frown}{\xi} < \xi_*, \quad \xi_* := q/(p+q), \\ 0 & \text{if } \xi > \xi_*, \\ -\delta^2/\varepsilon & \text{if } \xi = \xi_*. \end{cases} \tag{5.6}$$

is optimal and $W(\xi)$ has the form

$$W(\xi) = \begin{cases} p\xi\left(2 + \ln\dfrac{q(1-\xi)}{p\xi}\right) & \text{if } \xi \leq \xi_*, \\ q(1-\xi)\left(2 + \ln\dfrac{p\xi}{q(1-\xi)}\right) & \text{if } \xi \geq \xi_*. \end{cases} \tag{5.7}$$

In Case A, the strategy given by the synthesis

$$\alpha^*(\xi) := \begin{cases} 0 & \text{if } \xi > \xi_*, \quad \xi_* = r/(p+r), \\ 1 & \text{if } \xi < \xi_*, \\ \text{arbitrary} & \text{if } \xi = \xi_*, \end{cases} \tag{5.8}$$

is optimal and $W(\xi)$ has the form

$$W(\xi) = \begin{cases} p\xi\left[\ln\dfrac{r(1-\xi)}{p\xi} + 1 + \dfrac{\lambda_1^2\gamma^2\gamma^2 r}{2\varepsilon_2}\right] & \text{if } \xi \leq \xi_*, \\ \xi W_{(1)}(\xi) + (1-\xi)W_{(2)}(\xi) & \text{if } \xi \geq \xi_*. \end{cases} \tag{5.9}$$

where the formulae for the Laplace transformations of the functions obtained from $W_{(1)}(\xi)$ and $W_{(2)}(\xi)$ by a change of coordinates are given below (see (5.34),(5.35)). ∎

The statement of the theorem for Case B derives from Theorem 5.1, since for continuous time in this, and *only* in this, case the hypothesis matrix is a B-matrix. However, note that Theorem 5.3, given in the next section, gives an independent proof of Theorem 5.2 for Case B.

According to Theorem 4.4, to check that Theorem 5.2 holds in Cases A, C and D it suffices to show that the control $\alpha^*(\xi)$ and the function $W(\xi)$ in (5.6)–(5.9) satisfy the optimality equality (4.59) and

condition (4.60). The corresponding verifications will be made below, but for the sake of clarity we first present a general scheme to obtain the function $W(\xi)$ and calculate the value ξ_* in (5.6) and (5.8). Since the method given for finding the function $W(\xi)$ does not play any particular rôle here and is not used in the proof of the theorem, we shall give a heuristic discussion of the construction of the function $W(\xi)$. But first we shall reformulate some results from Chapter 4 in a form convenient for us.

$$* \qquad * \qquad *$$

According to (4.9) and (4.24), we consider for $\nu \leq \infty$ the maximization, with respect to all $\mathbb{F}$-predictable action rules $\beta = \beta(t)$, of the functional

$$W_\nu^\beta(\xi) := E_\xi^\beta \int_0^\nu [-\varepsilon_1\theta_1\beta_1(s) + \varepsilon_2\theta_2(1 - \beta(s))]\, ds$$

$$= E_\xi^\beta \int_0^\nu [-\varepsilon_1\xi(s)\beta(s) + \varepsilon_2(1 - \xi(s))(1 - \beta(s))]\, ds, \tag{5.10}$$

where the process $(\xi(t), 1 - \xi(t))$ satisfies equation (4.28).

By (4.4) the functional (5.10) can be rewritten as

$$W_\nu^\beta(\xi) = \xi E_1^\beta \int_0^\nu [-\varepsilon_1\beta(s)]\, ds + (1 - \xi)E_2^\beta \int_0^\nu \varepsilon_2(1 - \beta(s))\, ds. \tag{5.11}$$

As in discrete time, it is convenient to make a change of variables (*cf.* §3.4)

$$\tilde{\eta}(\xi) = \ln\frac{c\xi}{1 - \xi}, \quad \text{where} \quad c := \frac{-\varepsilon_1}{\varepsilon_2} = \frac{\lambda_1^2 - \lambda_1^1}{\lambda_2^1 - \lambda_2^2}. \tag{5.12}$$

Making the corresponding change of variables in (4.28) we obtain by elementary calculations that the process $\eta(t) := \tilde{\eta}(\xi(t))$ satisfies the equation

$$\eta(t) = \eta(0) - \sum_{j=1}^2 \left[-\gamma^j X^j(t) + \int_0^t \delta^j\beta^j(s)\, ds\right], \tag{5.13}$$

where $\beta^1(s) := \beta(s)$, $\beta^2(s) := 1 - \beta(s)$.

Let $\alpha(t, \eta)$ define a synthesis satisfying the equation

$$\frac{d\eta}{dt} = -\left[\delta^1 \alpha(t, \eta) + \delta^2(1 - \alpha(t, \eta))\right], \quad \eta(t_0) := \eta_0, \qquad (5.14)$$

i.e. the equation (5.14) is uniquely solvable in forward time for any $t_0 \geq 0$, $-\infty < \eta_0 < \infty$. If $\eta(t|t_0, \eta_0)$ is the solution of equation (5.14), then according to Lemma 4.2

$$\eta(s_-)I\{\tau_n < s \leq \tau_{n+1}\} = \eta(s|\tau_n, \eta(\tau_n))I\{\tau_n < s \leq \tau_{n+1}\}. \qquad (5.15)$$

In this case, $\beta(t) := \alpha(t, \eta(t))$, so that in the intervals between jumps of the process $X(t)$ the process $\eta(t)$ satisfies equation (5.14).

With the change of variable $\eta := \tilde{\eta}(\xi)$, the operator T^α of (4.56) becomes the operator $\tilde{T}^\alpha$ acting according to the formulae

$$\tilde{T}^\alpha f(t, \eta) = M^\alpha f(t, \eta) + (-\varepsilon_1)\tilde{\xi}(\eta)\alpha + \varepsilon_2(1 - \tilde{\xi}(\eta))(1 - \alpha)$$

$$M^\alpha f(t, \eta) := -\frac{\partial f(t, \eta)}{\partial t} \qquad (5.16)$$

$$+ \sum_{j=1}^{2}\left\{\left[f(t, \eta + \gamma^j) + f(t, \eta)\right]\tilde{p}^j(\eta) - \delta^j\frac{\partial f(t, \eta)}{\partial \eta}\right\}\alpha^j,$$

where $\alpha^1 := \alpha$, $\alpha^2 := 1 - \alpha$, $\tilde{\xi}(\eta) := e^\eta/(c + e^\eta)$ is the inverse transformation of $\tilde{\eta}(\xi)$ and finally

$$\tilde{p}^j(\eta) := p^j(\tilde{\xi}(\eta)) = \lambda_1^j\tilde{\xi}(\eta) + \lambda_2^j(1 - \tilde{\xi}(\eta)). \qquad (5.17)$$

Now Remark 4.5 and Theorem 4.6 can be reformulated as follows.

Lemma 5.1

(a) *Let the function $\alpha(t, \eta)$ be such that $\alpha_\nu(s, \eta) := \alpha(\nu - s, \eta)$, $(0 \leq t < \infty$, $0 < \nu < \infty$, $0 \leq s \leq \nu)$ defines a synthesis satisfying equation (5.14) and let there exist a continuously differentiable function $U(t, \eta)$ such that $U(0, \eta) \equiv 0$ and*

$$\tilde{T}^{\alpha(t, \eta)}U(t, \eta) = 0. \qquad (5.18)$$

Then $U(t, \tilde{\eta}(\xi)) = W_t^\beta(\xi)$, where β is the action rule corresponding to the synthesis $\alpha_\nu(s, \eta)$ with initial state ξ. If, moreover, the equality

$$\tilde{T}^{\alpha(t, \eta)}U(t, \eta) = \inf_\alpha \tilde{T}^\alpha U(t, \eta) = 0, \qquad (5.19)$$

holds, then $U(t, \tilde{\eta}(\xi)) = W_t(\xi)$ and $\alpha_\nu(s, \eta)$ defines an optimal strategy for the minimization of loss problem on the time interval $(0, \nu)$.

(b) *Let the function $\alpha(\eta)$ define a synthesis satisfying equation (5.14) and let there exist a continuously differentiable function $U(\eta)$ such that*

$$\tilde{T}^{\alpha(\eta)}U(\eta) = 0, \qquad (5.20)$$

$$E_\xi^\beta U(\tilde{\eta}(\xi(t))) \to 0 \quad \text{as} \quad t \to \infty, \qquad (5.21)$$

where β is the action rule corresponding to $\alpha(\eta)$ and initial point ξ. Then $U(\tilde{\eta}(\xi)) = W^\beta(\xi)$. If, moreover,

$$\tilde{T}^{\alpha(\eta)}U(\eta) = \inf_\alpha \tilde{T}^\alpha U(\eta) = 0 \qquad (5.22)$$

and (5.21) holds for any action rule $\alpha(\eta)$, then $U(\tilde{\eta}(\xi)) = W(\xi)$ and $\alpha(\eta)$ defines an optimal strategy on the infinite time interval $(0, \infty)$. ∎

*   *   *

Next we give a heuristic description of the construction of the function $W(\xi)$. We call a strategy an *a-threshold* strategy if it is given by a synthesis $\alpha^a(\eta)$ of the form

$$\alpha^a(\eta) = \begin{cases} 0 & \text{if } \eta > a \\ 1 & \text{if } \eta < a \\ \text{arbitrary} & \text{if } \eta = a \text{ in Case A} \\ -\delta^2/(\delta^1 - \delta^2) & \text{if } \eta = a \text{ in Cases C and D.} \end{cases} \qquad (5.23)$$

For $\eta = a$ the control α is chosen so that there exists a solution of equation (5.14). If π is an a-threshold strategy then the index π will be replaced by the index a as, for example, in $W^a(\xi)$, E_ξ^a and so on.

The above-mentioned construction scheme for $W(\xi)$ amounts briefly to the following: initially we show a method for finding the loss function $W^a(\xi)$ for an arbitrary a-threshold strategy and further for defining the value of the *threshold a_** which provides the minimal value of

loss amongst threshold strategies. This a_* and $W^{a_*}(\xi)$ appear in the formulation of Theorem 5.2.

For fixed ξ an arbitrary strategy π defines an action rule (depending on ξ) and according to (5.11) the value of the loss function $W^\pi(\xi)$ may be represented as

$$W^\pi(\xi) = \xi W_{(1)}^\pi(\xi) + (1 - \xi)W_{(2)}^\pi(\xi), \qquad (5.24)$$

where $W_{(i)}^\pi(\xi)$ is the value of the loss function under hypothesis H_i, $i = 1, 2$.

The constructions of the functions $U_1^\alpha(\eta) := W_{(1)}^a(\tilde\xi(\eta))$ and $U_2^a(\eta) := W_{(2)}^a(\tilde\xi(\eta))$ are conducted similarly, therefore we shall consider below mainly $U_1^a(\eta)$.

If the action rule corresponding to π is given by the synthesis $\alpha(\eta)$, then we have for $U_1^\pi(\eta)$ (see (5.11)) that

$$U_1^\pi(\eta) = E_{1,\eta}^\pi \int_0^\infty [-\varepsilon_1 \alpha(\eta(s))] \, ds. \qquad (5.25)$$

Note that the expectation in (5.25) corresponds to the measure P_1^π, but the process $\eta(s)$ corresponds to the measure P_ξ with $\eta(0) := \tilde\eta(\xi)$. If the function $U_1^\pi(\eta)$ is smooth, then (and similarly for the function $U^\pi(\eta) = W^\pi(\tilde\xi(\eta))$) it satisfies the equation

$$\sum_{j=1}^2 \{[U(\eta + \gamma^j) - U(\eta)] \lambda_1^j - \delta^j \frac{dU}{d\eta}(\eta)\} \alpha^j(\eta) - \varepsilon_1 \alpha^1(\eta) = 0, \quad (5.26)$$

obtained from equation (5.20) by the change of variables $\tilde\xi(\eta)$ for 1 and $(1 - \tilde\xi(\eta))$ for 0.

Similarly to Lemma 5.1, it can be shown that if $U(\eta)$ is a smooth solution of equation (5.26) satisfying the condition that $E_{1,\eta}^\pi U(\eta(t)) \to 0$ as $t \to \infty$, then $U(\eta)$ coincides with the value of the loss function for the synthesis $\alpha(\eta)$ under H_1.

We fix some a-threshold strategy $\alpha^a(\eta)$ and denote by $\tau := \tau_\eta^a$ the first time the process $\eta(t)$ hits the straight line $\eta = a$ in the (η, t) plane starting at the point η. Then, by the strong Markov property of the process $\eta(t)$, formula (5.25) can be rewritten as

$$U_1^a(\eta) = E_{1,\eta}^a \int_0^\tau [-\varepsilon_1 \alpha^a(\eta(s))] \, ds + E_{1,\eta}^a U_1^a(a) I\{\tau < \infty\}. \qquad (5.27)$$

If $\eta(s) \le a$ (correspondingly $\eta(s) > a$) for all $s \le \tau$, then (5.27) can be rewritten as

$$U_1^a(\eta) = -\varepsilon_1 \alpha(\eta) E_{1,\eta}^a \tau + U_1^a(a) P_{1,\eta}\{\tau < \infty\}, \qquad (5.28)$$

where $\alpha(\eta)$ is equal to 0 or 1 by (5.23), depending on whether η is greater or less than a.

From formulae (5.23) and (5.13) and from the definition of the process $X(t)$ it is easy to derive that under H_1 the distributions of the process $\eta(t)$ for $t \le \tau$ coincide with the distributions of the processes

$$\eta - \delta^1 t + \gamma^1 N^1(t) \qquad \text{if} \quad \eta < a$$
$$\eta - \delta^2 t + \gamma^2 N^2(t) \qquad \text{if} \quad \eta > a,$$

where $N^j(t)$ is a *Poisson process* with parameter λ_1^j, $j = 1, 2$.

Such processes have the form $\eta + gt + N(t)d$, where $N(t)$ is a Poisson process with parameter λ, $d := \ln(\lambda/(\lambda + g))$ which implies that $g + d\lambda \ge 0$. Writing the corresponding differential equations, it is simple to obtain (see, for example, Skorohod 1964, §26) that for the exit time τ at the point η on the straight line $\eta = a$,

$$P\{\tau < \infty\} = \exp(a - \eta) \text{ if } \eta \ge a, \, g < 0,$$

$$P\{\tau < \infty\} = 1, \, E\tau = (a - \eta)/(g + d\lambda) \text{ if } \eta < a, \, g > 0. \quad (5.29)$$

We now consider the construction of the function $U_1^a(\eta)$ in concrete cases.

(1) Cases C and D. In these cases formula (5.28) can be used. Then according to (5.29) we have

$$U_1^a(\eta) = \begin{cases} p(a - \eta) + U_1^a(a) & \text{if } \eta \le a, \\ U_1^a(a) \exp(a - \eta) & \text{if } \eta \ge a, \end{cases} \qquad (5.30)$$

where p is defined in (5.4). The constant $U_1^a(a)$ is found from equating left and right derivatives of the function $W_1^a(\eta)$ at the point a, which gives $U_1^a(a) = p$. Similarly,

$$U_2^a(\eta) = \begin{cases} q \, \exp(\eta - a) & \text{if } \eta \le a, \\ q(\eta - a) + q & \text{if } \eta \ge a. \end{cases} \qquad (5.31)$$

(2) Case A, $\eta \leq a$. Similarly to (1) we have that

$$U_1^a(\eta) = p(a - \eta) + U_1^a(a),$$

$$U_2^a(\eta) = U_2^a(a) \exp(\eta - a). \tag{5.32}$$

For $\eta > a$, formula (5.28) cannot be used, since in Case A, $\delta^2 < 0$, $\gamma^2 < 0$ and if the process $\eta(t)$ hits the straight line $\eta = a$, with probability 1 it hits it from below. We use the fact that $U_1^a(\eta)$ must be a solution of equation (5.26). For $\eta > a$, equation (5.26) has the form

$$\lambda_1^2 \left[U(\eta + \gamma^2) - U(\eta) \right] - \delta^2 \frac{dU(\eta)}{d\eta} = 0. \tag{5.33}$$

We shall find the solution of (5.33) which satisfies the additional condition $\lim_{\eta \to \infty} U(\eta) = 0$.

We apply Laplace transformation to equation (5.33). Defining $\tilde{U}(\lambda) := \lambda \int_0^\infty \exp(-\lambda s) U(a+s) \, ds$ and using the fact that the function $U_1^a(\eta)$ for $\eta < a$ is defined by (5.32), we obtain

$$\tilde{U}(\lambda) - U_1^a(a) = \frac{-\lambda \gamma^2 - 1 + \exp(\lambda \gamma^2)}{\lambda(\lambda \delta^2 + \lambda_1^2 - \lambda_1^2 \exp(\lambda \gamma^2))} \lambda_1^2 p_1. \tag{5.34}$$

The constant $U_1^a(a)$ is defined from the condition that $\lim_{\eta \to \infty} U_1^a(\eta) = 0$. Since this condition is equivalent to the equality $\tilde{U}(0) = 0$, then we put $(-1)U_1^a(a)$ equal to the right-hand side of (5.34) as $\lambda \to 0$, i.e. $\gamma^2 \cdot \gamma^2 \cdot \lambda_1^2 p / 2(\delta^2 - \lambda_1^2 \gamma^2)$.

The computation of $U_2^a(\eta)$ is conducted similarly with the exception that the expression for the Laplace transform has the form

$$\tilde{U}(\lambda) \left[-\lambda_2^2 - \delta^2 \lambda + \lambda_2^2 \exp(\lambda \gamma^2) \right] + \varepsilon_2$$

$$+ U_2^a(a) \left[\lambda \delta^2 + (\lambda/(\lambda - 1))(\lambda_1^2 - \lambda_2^2 \exp(\lambda \gamma^2)) \right] = 0. \tag{5.35}$$

The constant $U_2^a(a)$ is found from the equality to 0 of the limit of the left-hand side of (5.35) as $\lambda \to 1$, from which it follows that $U_2^a(a) = r$, where r is defined in (5.4).

Combining the formulae derived and taking into account that $U^a(\eta) = \tilde{\xi}(\eta) U_1^a(\eta) + (1 - \tilde{\xi}(\eta)) U_2^a(\eta)$, we have in Cases A, C and D that the value $U^a(\eta)$ for $\eta < a$ can be written as

$$U^a(\eta) = \frac{\exp \eta}{c + \exp \eta} \left[p(a - \eta) + U_1^a(a) + cQ \exp(-a) \right], \tag{5.36}$$

where $Q := q$ in Cases C and D and $Q := r$ in Case A, $U_1^a(a) := p$ in Cases C and D, $U_1^a(a) = \gamma^2 \cdot \gamma^2 \cdot \lambda_1^2 p / 2(\delta^2 - \lambda_1^2 \gamma^2)$ in Case A.

In Cases C and D, a formula similar to (5.36) holds for $\eta > a$, and it is easily checked that $\min_a U^a(\eta) = U^{a*}(\eta)$ holds for any η, where $a_* := \ln(cQ/p)$, i.e. in the class of threshold strategies the strategy with the threshold a_* will be optimal, which coincides in ξ coordinates with the value ξ_* in (5.36). In Case A, define $a_* := \ln(cQ/p)$. According to (5.36), for fixed $\eta < a_*$ the best strategy amongst all a-threshold strategies with $a > \eta$ is the a_*-threshold strategy.

$$* \qquad * \qquad *$$

We now show that the function $U^*(\eta) := U^{a*}(\eta)$ constructed satisfies equation (5.22) and relation (5.21) for any action rule β. From Lemma 5.1 the statement of Theorem 5.2 follows for Cases A, C and D. Equation (5.22) can be rewritten as

$$\inf_\alpha \left[\alpha K^1 + (1 - \alpha) K^2 \right] = 0,$$

where

$$K^j := K^j(\eta)$$

$$= \bar{p}^j(\eta) \left[U(\eta + \gamma^j) - U(\eta) \right] - \delta^j \frac{dU}{d\eta} + |\varepsilon_j| \tilde{\xi}^j(\eta),$$

$$\xi^1 := \xi, \quad \xi^2 := 1 - \xi. \tag{5.37}$$

In Cases C and D we substitute the formula obtained for $U^*(\eta)$ in the expressions for K^1 and K^2; it is seen directly that for $\eta < a_*$, $K^1 \equiv 0$, $K^2 > 0$ hold, and for $\eta > a_*$, $K^1 > 0$, $K^2 \equiv 0$ hold, therefore the functions $U^*(\eta)$ and $\alpha^*(\eta) := \alpha^{a*}(\eta)$ satisfy equation (5.22), and $d^2 U^*(\eta)/d\eta^2$ is continuous at the point $\eta := a_*$.

In Case A, similarly to Cases C and D, we obtain for $\eta < a_*$ that $K^1 \equiv 0$, $K^2 > 0$ and $\lim_{\eta \to a_*} K^2(\eta) = 0$. Using the representation (5.24) and the facts that $U_1^*(\eta)$ satisfies equation (5.33) and $U_2^*(\eta)$ satisfies the analogous equation, it is easily checked that $K^2 \equiv 0$ for $\eta > a_*$. Thus, $U^*(\eta)$ and $\alpha^*(\eta)$ satisfy equation (5.20) and $U^*(\eta)$ is continuously differentiable at the point a_*.

To prove the inequality $K^1 > 0$ for $\eta > a_*$, we use the fact that for $L(\eta) := K^1(\eta) - K^2(\eta)$, the formula

$$\delta^2 \frac{dL(\eta)}{d\eta} = \sum_{j=1}^{2} \tilde{p}^j(\eta)\alpha^j(\eta + \gamma^j)L(\eta + \gamma^j) - p^2(\eta)L(\eta) \qquad (5.38)$$

holds.

Indeed, as shown above, for $\eta > a_*$, $L(\eta) = K^1(\eta)$, $K^2(\eta) = 0$ hold. From the last equality and (5.37), we can express $dU/d\eta$ through the values of $U(\cdot)$ at the corresponding points. We substitute this expression in the definition of $K^1(\eta)$, differentiating and taking into account that in the corresponding regions either $K^1(\eta) = 0$ or $K^2(\eta) = 0$, and after easy calculations we obtain (5.38). A more detailed proof is given in §5.3, where a general formula is obtained holding for all $\nu < \infty$ (see (5.65)).

From the obtained relations $K^1(\eta) \equiv 0$, $K^2(\eta) > 0$ for $\eta < a_*$ and $K^1(\eta) > 0$, $K^2(\eta) \equiv 0$ for $\eta > \alpha_*$, formula (5.38) for $\eta \geq a_*$ can be rewritten as

$$\delta^2 \frac{dK^1(\eta)}{d\eta} = -\tilde{p}^1(\eta)K^2(\eta + \gamma^1) + \tilde{p}^2(\eta)\left[K^1(\eta + \gamma^2) - K^1(\eta)\right]. \quad (5.39)$$

By the continuous differentiability of $U^*(\eta)$, we have that $K^1(a_*) = 0$. From this and from (5.39) it is easy to see that $((dK^1)(d\eta))(\eta) > 0$, $K^1(\eta) > 0$ for $\eta > a_*$, which means that $U^*(\eta)$ satisfies equation (5.22).

To complete the proof it remains to show that (5.21) holds for $U^*(\eta)$. It suffices to check that in all the cases considered, A, C and D, the function $W(\xi) := U^*(\tilde{\eta}(\xi))$ satisfies

$$\lim_{\xi \to 0} W'^{a*}(\xi) = \lim_{\xi \to 1} W'^{a*}(\xi) = 0 \qquad (5.40)$$

and that

$$\xi(t) \to \theta(\omega) \quad P_\xi^\beta\text{--almost surely as } t \to \infty \qquad (5.41)$$

for any strategy β.

To check (5.40) is nontrivial only in Case A as $\xi \to 1$, by construction $W_{(1)}(\xi) \to 0$ as $\xi \to 1$, and from formulae (5.35) it is easily obtained that $U_2^*(\eta)$ grows linearly along η, which implies that $(1 - \xi)W_{(2)}(\xi) = (1 - \xi)U_2(\tilde{\eta}(\xi)) \to 0$ as $\xi \to 1$.

To prove (5.41), we write the process $\eta(t)\theta(\omega)$ using (5.13) as

$$\theta(\eta(t) - \eta(0)) = -\theta \sum_{j=1}^{2}(\delta^j - \lambda_1^j \gamma^j)\int_0^t \beta^j(s)\,ds$$
$$+ \theta \sum_{j=1}^{2} \gamma^j(X^j(t) - \lambda_1^j \int_0^t \beta^j(s)\,ds. \qquad (5.42)$$

But the processes $B_t^j := \theta\gamma^j\left[X^j(t) - \lambda_1^j \int_0^t \beta^j(s)\,ds\right]$, $j = 1, 2$, are orthogonal martingales according to (4.5) and therefore

$$E_\xi^\beta\left[B_t^1 + B_t^2\right]^2$$
$$= \sum_{j=1}^{2}(\gamma^j)^2\lambda_1^j E_\xi^\beta\left[\theta \int_0^t \beta^j(s)\,ds\right] \leq t\sum_{j=1}^{2}(\gamma^j)^2\lambda_1^j. \qquad (5.43)$$

Since $\lambda_2^j/\lambda_1^j \neq 1$ in Cases A, C and D, therefore

$$\min_j\left[\delta^j - \lambda_1^j\gamma^j\right]$$
$$= \min_j\left[\lambda_1^j(1 - \lambda_2^j/\lambda_1^j + \ln(\lambda_2^j/\lambda_1^j)\right] = -a < 0. \qquad (5.44)$$

Using now in sequence (5.42), (5.44), Chebychev's inequality and (5.43) we have

$$P_\xi^\beta\{\theta(\eta(t) - \eta(0)) < \theta at/2\}$$
$$= P_\xi^\beta\left\{B_t^1 + B_t^2 < \theta\left[at/2 + \sum_{j=1}^{2}(\delta^j - \gamma^j\lambda_1^j)\int_0^t \beta^j(s)\,ds\right]\right\}$$
$$\leq P_\xi^\beta\{B_t^1 + B_t^2 < -\theta at/2\} \leq P_\xi^\beta\{|B_t^1 + B_t^2| > at/2\}$$
$$\leq 4E_\xi^\beta(B_t^1 + B_t^2)^2/(at)^2 < c/t,$$

which implies that from $\theta = 1$ it follows that $\eta(t) \to \infty$, i.e. $\xi(t) \to 1$ almost surely with respect to the measure P_ξ^β as $t \to \infty$. By considering $(1 - \theta)(\eta(t) - \eta(0))$, we similarly obtain that from $\theta = 0$ it follows that $\xi(t) \to 0$ as $t \to \infty$ almost surely with respect to the measure P_ξ^β, which completes the proof of the statement of Theorem 5.2 for Cases A, C and D.

$$* \qquad * \qquad *$$

Consider now Case B. In §5.3, we prove independently Theorem 5.3 which states the existence in Case B of an increasing continuous function $l(t)$ such that $l(0) = 0$, $l(t) \to \infty$ as $t \to \infty$ and the optimal strategy in the problem on the time interval $(0, \nu)$ is given by the synthesis $\alpha(s, \eta) := \alpha^*(\nu - s, \eta)$, where $\alpha^*(t, \eta) = 1$ for $\eta \leq l(t)$, $\alpha^*(t, \eta) = 0$ for $\eta > l(t)$.

For any k and η, let $t_0 := k/\varepsilon_1 \xi(\tilde{\eta})$, define s_0 from the condition $l(s_0) = \eta + |\delta^1| t_0$ and assume that $\nu > t_0 + s_0$.

Denote by $\eta(t)$ the process (5.13) on the interval $(0, \nu)$ corresponding to the action rule β^* which is given by the synthesis $\alpha(t, \eta)$ for the initial point η. Let $\tau := \min[\nu, \min_{0 \leq s \leq \nu}\{s : \eta(s) \geq l(\nu - s)\}]$. For $t < \tau$, $\beta^*(t) \equiv 1$ holds, which means that $X^2(t) = 0$ and by (5.13), $\eta(t) \leq \eta + |\delta^1| t$.

Suppose that $\tau < t_0$. Then

$$\eta(\tau) \leq \eta + |\delta^1|\tau < \eta + |\delta^1| t_0 = l(s_0) \leq l(\nu - t_0) \leq l(\nu - \tau)$$

and the obtained contradiction shows that $\tau \geq t_0$. Therefore, from (5.10) we have that

$$W_\nu(\tilde{\xi}(\eta)) \geq |\varepsilon_1| E_{\tilde{\xi}(\eta)}^{\beta^*}\left[\int_0^\tau \xi(s)\beta^*(s)\, ds\right]$$
$$= |\varepsilon_1| E_{\tilde{\xi}(\eta)}^{\beta^*} \theta\tau \geq |\varepsilon_1|\tilde{\xi}(\eta) t_0 = k.$$

Since k is arbitrary, it follows that $W_\nu(\xi) \to \infty$ as $\nu \to \infty$, and this completes the proof of Theorem 5.2. ∎

5.3 Minimization of loss over a finite horizon for the case $m = N = 2$

In §1.12 we discussed the optimal synthesis for the problem of loss minimization over a finite time horizon ν as follows. There exists a curve $l(t)$ dividing the half plane $\{(t, \eta) : t \geq 0\}$ into two parts. If the time remaining equals t, then the optimal strategy prescribes use of the first device if $\tilde{\eta}(\xi(\nu - t)) < l(t)$, and of the second if $\tilde{\eta}(\xi(\nu - t)) > l(t)$. On the curve $l(t)$, the control is chosen so that a solution of equation (5.14) exists and the constant c in (5.12) is chosen so that the equality $l(0) = 0$ holds. We formulate here the precise statement, assuming as before that (5.2) holds and $\varepsilon_1 \varepsilon_2 < 0$ and defining $l' := dl(t)/dt$.

We will say that *condition C_{t_*}* is fulfilled for the function $\eta = l(t)$, $t \geq 0$, if $l(0) = 0$, $l(t)$ is smooth and $\delta^2 < l'(t) < \infty$ on $0 < t < t_*$, $l(t)$ is piecewise smooth and $\delta^1 < l'(t) < \delta^2$ on $t_* < t < \infty$.

Theorem 5.3 *There exists a t_* $(0 \leq t_* \leq \infty)$ and a function $l(t)$ which satisfies condition C_{t_*} such that any optimal action rule in the problem of loss minimization over the time horizon ν is given by the synthesis $\alpha(t, \eta) := \alpha^*(\nu - t, \eta)$, where*

$$\alpha^*(t, \eta) = \begin{cases} 1 & \text{if } \eta < l(t) \\ 0 & \text{if } \eta > l(t) \\ (l'(t) - \delta^2)/(\delta^1 - \delta^2) & \text{if } \eta = l(t), \quad t > t_* \\ \text{arbitrary} & \text{if } \eta = l(t), \quad 0 \leq t \leq t_*. \end{cases} \qquad (5.45)$$

Here:

(a) in Case D, the curve $\eta = l(t)$ is a turnpike, i.e. $t_* = 0$,

(b) in Cases A and B, the curve is a switching line, i.e. $t_* = \infty$,

(c) in Case C, part of the curve is a turnpike and part is a switching line, i.e. $0 < t_* < \infty$. ∎

Remark 5.1 In the case of symmetric hypotheses, it is easy to derive from the proof given below that $t_* = 0$ and $l(t) \equiv 0$. The same result will be obtained independently in §6.5. ∎

Proof. To prove the theorem we use Lemma 5.1, but it is convenient to consider the problem of maximization of the number of successes rather than the problem of minimization of loss. The theorem will be proved if we construct t_* and $l(t)$ satisfying C_{t_*} and the continuously differentiable function $U(t, \eta)$ such that $U(0, \eta) = 0$ and $U(t, \eta)$ and $\alpha^*(t, \eta)$ from (5.45) satisfy the optimality equation (5.19). For the problem of maximization of the number of successes this equation has the form

$$\frac{\partial U(t, \eta)}{\partial t} = \sup_{\alpha} \left[\alpha B^1 U(t, \eta) + (1 - \alpha) B^2 U(t, \eta) \right], \qquad (5.46)$$

where

$$B^j U(t, \eta) := \tilde{p}^j(\eta) \left[1 + U(t, \eta + \gamma^j) - U(t, \eta) \right] - \delta^j \frac{\partial U(t, \eta)}{\partial \eta},$$
$$j = 1, 2.$$

If we define

$$L(t, \eta) := B^1 U(t, \eta) - B^2 U(t, \eta), \qquad (5.47)$$

then from the above it follows that it suffices to construct $l(t)$ and $U(t, \eta)$ such that

$$\frac{\partial U(t, \eta)}{\partial t} = B^j U(t, \eta) \qquad \text{if} \quad (t, \eta) \in A^j \qquad (5.48)$$

$$(-1)^j L(t, \eta) < 0 \qquad \text{if} \quad (t, \eta) \in A^j, \quad j = 1, 2, \qquad (5.49)$$

where

$$A^1 := \{(t, \eta) : t \geq 0, \ \eta < l(t)\}, \qquad A^2 := \{(t, \eta) : t \geq 0, \ \eta > l(t)\}.$$

In the general case it is unrealistic to expect to obtain an explicit expression for $l(t)$ and $U(t, \eta)$, and therefore only an algorithm for their iterative construction will be given. In the lemmas given below it will be proved that this algorithm can be realized at each step. These lemmas are proved for Cases A and B; in Case C, only the existence of $t_* < \infty$ and the possibility of constructing $l(t)$ and $U(t, \eta)$ for $t \leq t_*$ are proved, which yields optimality in this region.

* \qquad * \qquad *

We consider first the case $\delta^2 \geq 0$. Before we give the algorithm, we assume given some curve $\eta = l(t)$ satisfying the condition C_{t_*} and then show that for this $l(t)$ the general solution of equation (5.48) can be found.

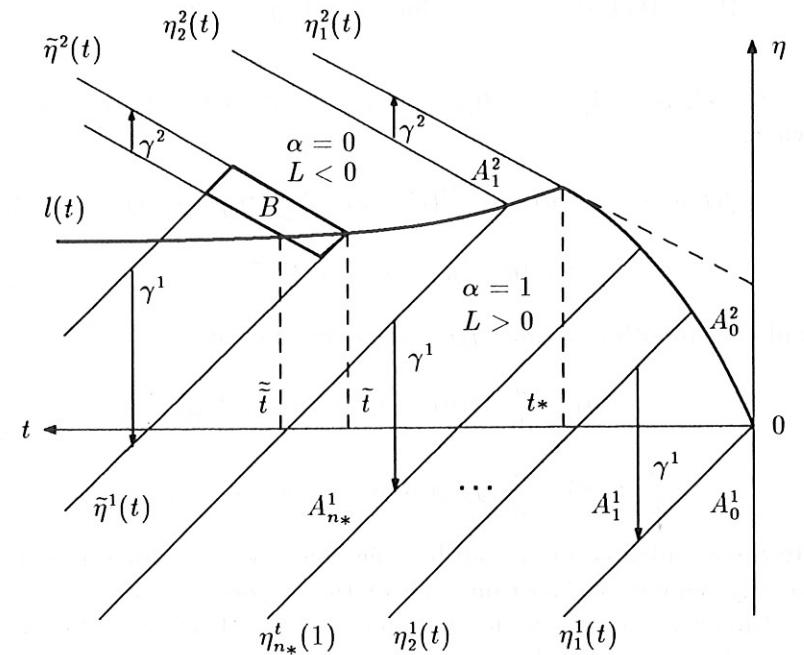We introduce the regions A_n^j $(j = 1, 2, \ n = 0, 1, \ldots)$ as follows (see Figure 9).

Figure 9

Optimal synthesis for the problem of loss minimization in continuous time with $m = N = 2$ (in coordinates η, t, where t is the time remaining) for $\delta^2 \geq 0$.

Define for $n = 1, 2, \ldots$

$$\eta_n^1(t) := \delta^1 t - (n - 1)\gamma^1, \qquad \eta_n^2(t) := \delta^2(t - t_*) + l(t_*) - (n - 1)\gamma^2$$