Further, let the initial point $j_0, t_0 := 0$, $\xi_0 := \xi$ and some strategy π be fixed. As mentioned in §2.4, for the discrete time problem for a fixed initial distribution we can find for each strategy at least as good a nonrandomized strategy, such that for each $n = 1, 2, \dots$ it defines a measurable mapping $\alpha_n(\cdot|h)$ from the space $\mathcal{H}_{n-1}$ into $\mathcal{A}$. Therefore the strategy π may be considered to be nonrandomized. We construct an action rule such that the cost of the strategy π for the initial point $j_0, t_0 := 0$, $\xi_0 := \xi$ coincides with $F^{\beta}(\xi)$ for any functional of type (4.48). For this the sequence of Borel functions $\{\beta_n(j_{1,n-1}, t_{1,n-1}, s),\ n = 1, 2, \dots\}$ given by the equalities $\beta_1(s) := \alpha_1(s|j_0, t_0, \xi_0)$, $\beta_2(j_1, t_1, s) := \alpha_2(s|j_0, j_1, t_0, t_1, \xi_0, \xi_1(t_1|\xi_0, t_0, \alpha_1(\cdot|j_0, t_0, \xi_0)), \alpha_1(\cdot|j_0, t_0, \xi_0))$, and so on, are sufficient. The theorem is proved. ■

Now it is possible to make the same remark as in §2.4 about the correspondence between action rules in the initial problem and nonrandomized strategies in the equivalent problem with complete information. It is convenient, however, to have a definition of a strategy in the initial problem as a function of the *a posteriori* probabilities.

Nonrandomized strategies in the problem with complete information may be assumed to be strategies of type $\{\alpha_n(s|j_{0,n-1}, t_{0,n-1}, \xi_{0,n-\nu},\ n = 1, 2, \dots\}$ since the controls may be sequentially excluded from the history $h = \{j_{0,n-1}, t_{0,n-1}, \xi_{0,n-1}, \alpha_{1,n-1}(\cdot)\}$. By the correspondence stated in the proof of Theorem 4.3, we will also call such a sequence a *strategy* for the initial problem. For this strategy we will use the notation $F^{\pi}_{\nu}(\xi)$ ($W^{\pi}_{\nu}(\xi)$). This notation is appropriate since for given ξ an action rule corresponds to each strategy.

* * *

The situation when the matrix $Q_n(\cdot)$ in (4.50) (or, equivalently, the function $\phi^j_{i,n}(\cdot)$ in (4.48)) does not depend on $j_{1,n-2}, t_{1,n-1}$, i.e. for $n \geq 2$ has the form $Q_n(j, t, s)$, is of special interest. In this case (which we will call the *Markov case*) the formulated discrete problem is a Markov model (see §2.4). If $Q_n(j, t, s)$ does not depend on j, then it is obvious that in the corresponding discrete model the j may be excluded from the state space. In continuous time, this corresponds to controlling only by the process $\xi(t)$ rather than the pairs $X(t)$, $\xi(t)$. For simplicity, only such a situation will be considered further.

In the Markov case, a question naturally arises regarding sufficiency of the class of Markov strategies and the existence of Markov uniformly optimal strategies (a repetition of the questions raised in §2.4). As usual, the answer to these questions is obtained from a study of the optimality equation. In spite of the fact that the corresponding theorem holds for arbitrary functions $Q_n(j, t, s)$ which are bounded from below and nonnegative for $\nu = \infty$, we give a formulation and a proof only for the case when $Q_n(\cdot)$ does not depend on j and t for each n and, for the case $\nu < \infty$, depends only on $\nu - s$, and in the case $\nu := \infty$ depends only on s.

Let $F^{\pi}_n(t, \xi) := F^{\pi}_{n\infty}(t, \xi)$ be the cost of the strategy π on the interval $[n, \infty)$ for the Markov model corresponding to the discrete time problem formulated at the beginning of this section.

Let π be some nonrandomized strategy in the model with discrete time on the time interval $[n, \infty)$ and let $\beta := \beta(n)$ be an action rule in the problem with continuous time, which after time τ_n coincides with the strategy π for initial point $\xi(\tau_n)$. Using the definition of the model and the results of Lemma 4.2, we obtain that, independent of the dependency on the values $\beta(s)$ up to τ_n,

$$E^{\beta}_{\xi}\left[\int_{\tau_n}^{\infty} \xi(s)Q(s)\beta^*(s)\,ds|\mathcal{F}_{\tau_n}\right] = F^{\pi}_n(\tau_n, \xi(\tau_n)). \qquad (4.51)$$

Let $F_n(t, \xi) := \inf_{\pi} F^{\pi}_n(t, \xi)$. From Theorem 4.4 and also from formula (4.51) it follows that

$$F_0(0, \xi) = F(\xi) = \inf_{\beta \in \Pi} F^{\beta}(\xi), \qquad (4.52)$$

where $F^{\beta}(\xi)$ is defined according to (4.48).

According to §2.4 and to the definition of the transition probabilities (4.49) and the cost function (4.50), the *optimality equation* for the function $F_n(t, \xi)$ has the form

$$\begin{aligned} F_{n-1}(t, \xi) &= \inf_{\alpha(\cdot) \in \mathcal{A}} T^{\alpha(\cdot)} F_n(t, \xi) \\ &:= \inf_{\alpha(\cdot) \in \mathcal{A}} \left[\int_t^{\infty} [\xi(s|a)Q_n(s)\alpha^*(s)z(s|a) + \right. \\ &\qquad \left. \sum_{l=1}^{m} F_n(s, \Gamma^l \xi(s|a))\alpha^l(s)p^l(\xi(s|a))z(s|a)]\,ds\right]. \end{aligned} \qquad (4.53)$$

Theorem 4.5 *For $\nu < \infty$ always, and for $\nu = \infty$ when all elements of matrix $Q_n(s)$ are nonnegative, the functions $F_n(t,\xi)$ satisfy the optimality equation and there exists a Markov uniformly optimal strategy which in the uniform model (i.e. when $Q_n(s)$ does not depend on n) may be chosen to be stationary.*

Proof. The statement of the theorem coincides with the statement of Theorem 2.7. However, the latter was proved only for a finite number of controls. Now for the proof of the corresponding result we will use the fact that for the case of an infinite time interval for a *maximization* problem the following conditions are sufficient: (a) the profit function is nonpositive, (b) the model is upper semicontinuous (see Dynkin & Yushkevitch 1976), (c) the optimal value of the functional is upper semicontinuous with respect to the initial point. (This statement was proved (Dynkin & Yushkevitch, §5.6) for the case when condition (b) is replaced by the requirement that the model be bounded from below (which is not relevant to the problem considered by us), but in this work it was also stated that the boundedness is needed only to prove that condition (c) is true.)

Since we are considering a minimization problem, (a) must be replaced by the nonnegativity of the cost function and in conditions (b) and (c) upper semicontinuity must be replaced by lower semicontinuity.

By assumption of the theorem and Remark 4.2 it may be assumed that the cost function is nonnegative.

We turn to the proof that the changed condition (b) holds. By the compactness of the set $\mathcal{A}$ it suffices to ensure that the transformation $T^{\alpha(\cdot)}$ (see (4.53)) transforms functions which are lower semicontinuous with respect to t and ξ to functions which are lower semicontinuous with respect to $a := (t, \xi, \alpha(\cdot))$.

Suppose (without loss of generality) that $\xi_N \neq 0$. Making the change of variables $\eta_i(s) := \ln(\xi_i(s)/\xi_N(s))$, $i = 1, \ldots, N$, we have that $\eta_i(s)$ satisfies the equation (see (1.12))

$$\frac{d\eta_i(s|a)}{ds} = -\sum_{j=1}^{m}(\lambda_i^j - \lambda_N^j)\alpha^j(s), \quad i = 1, \ldots, N. \tag{4.54}$$

From this it follows that $\eta(s|a)$, and hence $\xi(s|a)$ and $z(s|a)$ are lower semicontinuous with respect to a uniformly with respect to s on each finite time interval. If in the definition of the transformation $T^{\alpha(\cdot)}$ we consider the interval from t to ν and replace $Q_{n+1}(\xi)$ with 0 when $Q_{n+1}(\xi) \geq M$, then we obtain the lower semicontinuity with respect to a of the "truncated" transformation $T^{\alpha(\cdot)}$. Taking the limit with respect to both ν and M, we obtain the lower semicontinuity of the transformation $T^{\alpha(\cdot)}$ with respect to a.

To check condition (c) we note the following. From (4.52) and Remark 4.3 it follows that $F_0(0,\xi)$ may be considered to be a solution of the finite Bayesian problem. Exactly similarly, for fixed $n > 0$ and $t \geq 0$ the function $F_n(t)$ may be considered to be a solution of the finite Bayesian problem (with initial time t, *a priori* distribution ξ and functions $\tilde{Q}_r(s)$, $r = 1, 2, \ldots$ coinciding with the functions $Q_{r+n}(s)$). Therefore, by Lemma 2.3, $F_n(t,\xi)$ is convex with respect to ξ, continuous on the interior of a simplex S^N and its restriction to the interior of any face of any dimension is also continuous. This implies that $F_n(t,\xi)$ is lower semicontinuous with respect to ξ. Further, if $Q_n(s)$ is replaced by 0 for $n \geq \nu$, $s \geq \nu$ and $Q_n(s) \geq M$ and some action rule β is fixed, then $F_n^\beta(t,\xi)$ will be continuous with respect to t uniformly with respect to β and ξ. (This follows from the boundedness of $Q_n(s)$ and the continuity with respect to a of functions $\xi(s|a)$ and $z(s|a)$ uniformly with respect to s on each finite interval proved above (see (4.54)). But then $F_n(t,\xi) := \inf_\beta F_n^\beta(t,\xi)$ will be a continuous function of t uniformly with respect to ξ. From this and continuity with respect to ξ follows continuity with respect to both t and ξ of the "truncated" functions $Q_n(s)$ (for ξ belonging to the interior of the simplex or to the interior of any face). Letting n_0, ν and M tend to infinity and using the compactness of the randomized action rules in the problem with continuous time, as in the proof of Theorem 4.3, we obtain monotonic convergence to $F_n(t,\xi)$. Thus, $F_n(t,\xi)$ is lower semicontinuous with respect to both (t,ξ) as a limit of a nondecreasing sequence of lower semicontinuous functions. So Theorem 4.5 is proved. ■

* * *

There exist different approaches to the solution of the optimality equation and to finding an optimal strategy. For example, it may be possible from the differential equations (4.28) to express the functions

$\xi(s|a)$ and $z(s|a)$ in a closed form in terms of the control $\alpha(\cdot)$ and, replacing the expressions obtained into (4.52), we can then find the control $\alpha(\cdot)$ giving an infimum recursively. However, such an approach is successful only in simple cases. In §7.3 the problem of maximizing the probability of the event of at least one jump in a fixed time interval, i.e. the problem with $\phi^j_{i,1} := -1$, $\phi^j_{i,n} := 0$ for $n > 1$ for all i and j, is treated with this approach.

Another possible approach consists in the following.

For some n, suppose the functions $F_{n+1}(t,\xi)$ are already known, and that $F_{n+1}(t,\xi)$ is continuously differentiable. Then according to (4.52) and (4.28) the problem of finding $F_n(t,\xi)$ may be considered as a nonautonomous problem of Pontryagin type with state variables ξ and z with fixed left end point and an integral functional. Such an approach for the case $m = N = 2$ will be discussed in Chapter 6. In particular, the situation described obtains in the problem of maximizing the probability of the appearance of at least k jumps up to the fixed time ν. The corresponding functional is obtained for $\phi^j_{ik}(s) := -1$ for $0 \le s \le \nu$, $\phi^j_{in}(s) := 0$ for $n \ne k$.

Finally, one more possible approach consists in obtaining the optimal strategies by "analysing" the optimality equation. This will be the subject of the next section.

4.5 Local optimality equation and optimal synthesis

In the initial problem it was required for a fixed initial distribution ξ to find an action rule (i.e. to demonstrate a sequence $\beta = \{\beta_m(j_{1,n-1}, t_{1n}),\ n = 1,\dots\}$) such that the measure P^β_ξ induced on $\mathcal{F}^\theta_\infty$ by the action rule β minimizes the functional $F^\beta_\nu(\xi)$ (see (4.11) and (4.12)).

In §4.2 the solution to this problem was reduced to the solution of a problem of control with complete information involving the control of the pair of processes $X(t)$, $\xi(t)$ and the criterion functional (4.24). In §4.4 it was shown that such a problem may also be considered as a control problem in discrete time for which the idea of a strategy arises as well as the idea of an action rule. From a nonrandomized strategy (an optimal strategy may be chosen from this class) for a fixed *a priori* distribution ξ on the set of hypotheses an action rule for the initial problem may be constructed.

In this section we consider only the Markov case, where the functional (4.24) given by the functions $Q_n(j_{1,n-1}, t_{1,n-1}, s)$ (see (4.50)) has the form $Q_n(j_{n-1}, t_{n-1}, s)$. Moreover, for simplicity we assume that $Q_n(\cdot)$ does not depend on j_{n-1}. In this case, instead of the pair $X(t)$, $\xi(t)$ it suffices to consider only the process $\xi(t)$ and we may restrict ourselves to strategies of the type $\pi := \{\alpha_n(s|t,\xi),\ n = 1,2,\dots\}$, where for fixed $n = 1,2,\dots$, $t \ge 0$ and $\xi \in S^N$ the function $\alpha_n(s|t,\xi)$ takes values in $\mathcal{A}$ and $\alpha_n(s|t,\xi)$ depends on t and ξ in a measurable way.

On the intervals between the jumps of the process the *a posteriori* probabilities satisfy the ordinary differential equation (4.28) (see Lemma 4.2), and similarly to the situation for deterministic optimal control problems the question arises whether the function $\alpha_n(s|t,\xi)$ may be given by some function of the time s and the *current* value of the *a posteriori* probability at time s.

Precisely, let there exist a function $\tilde{\alpha}(t,\xi)$ measurable with respect to t and ξ with values in S^N, such that the differential equation (4.28) for any initial point is uniquely solvable (in forward time) with the replacement of $\alpha(s)$ by $\tilde{\alpha}(s,\xi(s))$. Let $\tilde{\xi}(s|t,\xi)$ be the corresponding solution for the initial point t,ξ. If the equality $\alpha(s|t,\xi) = \tilde{\alpha}(s,\tilde{\xi}(s|t,\xi))$ holds, we will say that the function $\tilde{\alpha}(t,\xi)$ defines a *synthesis* for the controls $\alpha(s|t,\xi)$.

To address the question of the existence of an optimal synthesis and to find this synthesis, the local optimality equation may be used (see formula (4.57) below). A heuristic derivation of this equation is contained in Chapter 1. It can also be derived by differentiating the equality (4.53) along a trajectory of the process $\xi(t)$. The local optimality equation has the simplest form in the homogeneous case, but for the problem in continuous time the definition of homogeneity needs to be given more precisely. Let us say that a Markov problem is *homogeneous with respect to jumps* if its matrix functions $Q_n(j,t,s)$ do not depend on n. In this case the corresponding problem with discrete time is homogeneous and the functions $F_n(t,\xi)$ and the optimal controls $\alpha_n(s|t,\xi)$ will also not depend on n.

If for $\nu < \infty$ the functions $Q_n(\cdot)$ depend only on the difference $\nu - s$ and for $\nu := \infty$ the functions $Q_n(\cdot)$ do not depend on s, then in this case we will say that the problem is *homogeneous with respect to* *time.* In this case for $\nu = \infty$ the functions $F_n(t, \xi)$ will not depend on t and for $\nu < \infty$ will depend only on the difference $\nu - t$. As in the case of discrete time, for $\nu < \infty$ in the time homogeneous problem it is more convenient to consider the time remaining and, by changing notation, to consider that the function $F_n(t, \xi)$ gives the solution of the problem on any time interval of the type (t', ν), where $\nu - t' = t$.

If a problem is homogeneous in *both* time and jumps then we will speak simply about a *homogeneous problem.* In this case $Q(s)$ is a deterministic function and s is understood as time remaining for $\nu < \infty$.

In the homogeneous problem an expression similar to formula (4.51) may be proved, viz.

$$F(\nu - t, \xi) = \inf_\beta E^\beta_{\xi_0} \left[\int_t^\nu \xi(s) Q(\nu - s) \beta^*(s) ds | \xi(t) = \xi \right] \qquad (4.55)$$

with an obvious change for the case $\nu := \infty$.

Now we will formulate the statement of the *local optimality equation* for the *homogeneous case.*

We introduce the operator $T^\alpha (\alpha \in S^m)$ on continuously differentiable functions $f(t, \xi)$ by the formula

$$T^\alpha f(t, \xi) := -\frac{\partial f(t, \xi)}{\partial t} - (\text{grad}_\xi f(t, \xi)) g(\xi) \alpha^* + \sum_{l=1}^m \left[f(t, \Gamma^l \xi) - f(t, \xi) \right] p^l(\xi) \alpha^l + \xi Q(t) \alpha^*. \qquad (4.56)$$

Theorem 4.6

(a) *Let $\nu < \infty$ and suppose that there exists a continuously differentiable function $\phi(t, \xi)$ such that*

$$\inf_\alpha T^\alpha \phi(t, \xi) = 0, \qquad \phi(0, \xi) = 0. \qquad (4.57)$$

Then

$$\phi(\nu - t, \xi) \le \inf_\beta E^\beta_{\xi_0} \left\{ \int_t^\nu \xi(s) Q(\nu - s) \beta^*(s) ds | \xi(t) = \xi \right\}. \qquad (4.58)$$

If the function $\alpha^*(t, \xi)$ achieves the infimum in (4.57) and is such that $\alpha^*(\nu - t, \xi)$ gives a synthesis satisfying equation (4.28), then equality holds in (4.58) (in particular, $F_\nu(\xi) = \phi(\nu, \xi)$) and $\alpha^*(\nu - t, \xi)$ defines an optimal strategy.

(b) *Let $\nu = \infty$ and suppose that there exists a continuously differentiable function $\phi(\xi)$ such that*

$$\inf_\alpha T^\alpha \phi(\xi) = 0 \qquad (4.59)$$

and for any action rule β that

$$E^\beta_\xi \phi(\xi(t)) \to 0 \quad \text{as} \quad t \to \infty. \qquad (4.60)$$

Then

$$\phi(\xi) \le \inf_\beta E^\beta_{\xi_0} \left\{ \int_t^\infty \xi(s) Q(s) \beta^*(s) ds | \xi(t) = \xi \right\}. \qquad (4.61)$$

If the function $\alpha^*(\xi)$ achieves the infimum in (4.59) and gives a synthesis satisfying equation (4.28), then equality holds in (4.61) (in particular $F(\xi) = \phi(\xi)$) and $\alpha^*(\xi)$ defines an optimal strategy.

Proof. Let the action rule $\beta := \beta(s)$ be fixed. For the continuous differentiable function $\phi(\nu - t, \xi)$, the change of variables formula holds (see §9 of the Appendix):

$$\phi(\nu - v, \xi(v)) = \phi(\nu - t, \xi(t)) + \int_t^v \left[\frac{\partial}{\partial s} \phi(\nu - s, \xi(s)) - (\text{grad}_\xi \phi(\nu - s, \xi(s)) \cdot g(\xi(s)) \beta^*(s) \right] ds + \sum_{l=1}^m \int_t^v \left[\phi(\nu - s, \Gamma^l \xi(s_-)) - \phi(\nu - s, \xi(s_-)) \right] dX^l(s) \qquad (4.62)$$

at $t \le v \le \nu < \infty$. Note that for any predictable vector-valued function $\phi(s)$ we have that

$$E^\beta_{\xi_0} \left[\int_t^v \phi(s) dX^*(s) | \mathcal{F}_t \right] = E^\beta_{\xi_0} \left[\int_t^v \phi(s) [\text{diag}\, \beta(s)] p*(\xi(s))\, ds | \mathcal{F}_t \right].$$

Taking the conditional expectation with respect to $\xi(t)$ in (4.62) and then adding and subtracting the value $\int_t^v \xi(s) Q(\nu - s) \beta^*(s)\, ds$ within the expectation operation, we obtain

$$\begin{aligned}\phi(\nu - t, \xi) &= E^{\beta}_{\xi_0}\left[\phi(\nu - v, \xi(v)) | \xi(t) = \xi\right] \\ &\quad - E^{\beta}_{\xi_0}\int_t^v T^{\beta}\phi(\nu - s, \xi(s))ds \\ &\quad + E^{\beta}_{\xi_0}\left[\int_t^v \xi(s)Q(\nu - s)\beta^*(s)ds | \xi(t) = \xi\right].\end{aligned}$$

For $\nu < \infty$, set $v := \nu$ and by (4.57) we obtain (4.58). For $\nu = \infty$ instead of $Q(\nu - s)$ we need to take $Q(s)$ and let v tend to infinity. Then (4.61) follows from (4.59) and (4.60). If $\alpha^*(\nu - t, \xi)$ (correspondingly $\alpha^*(\xi)$) is a synthesis for the strategy defined by the function $\alpha^*(s|t, \xi)$, then for the action rule in (4.58) (correspondingly in (4.61)) corresponding to this strategy, we will achieve equality, which completes the proof of the theorem. ■

Remark 4.4 Statement (a) is the analogue of Theorem 2.5 for discrete time and (b) is the analogue of Theorem 2.6. The requirement in (b) that (4.60) must hold for *any* action rule β may be weakened by formulating conditions similar to (b) and (c) of Theorem 2.6. ■

Remark 4.5 Similarly to Theorem 4.6 it may be shown that if $\overline{\alpha}(\xi)$ gives a synthesis satisfying equation (4.28), $\overline{\beta}$ gives the action rule corresponding to this synthesis and that if there exists a continuously differentiable function $\overline{\phi}(\xi)$ such that

$$T^{\overline{\alpha}(\xi)}\overline{\phi}(\xi) = 0, \quad E^{\overline{\beta}}_{\xi}\overline{\phi}(\xi(t)) \to 0 \quad \text{as} \quad t \to \infty,$$

then $\overline{\phi}(\xi) = \overline{F}^{\beta}(\xi)$. The corresponding statement holds also for $\nu < \infty$. ■

5 SOLUTIONS OF SOME PROBLEMS IN THE BASIC CONTINUOUS TIME SCHEME

In this chapter, using the ideas and methods introduced in Chapter 4, we will study the problem of loss minimization in the continuous time case. In §5.1 we give without proof Theorem 5.1, describing the behaviour of the loss function at infinity, which is similar to Theorem 3.1, proved in §3.2, concerning the discrete time case. In §5.2 a complete solution is given for the problem of loss minimization over an infinite time horizon for second order matrices with $m = N = 2$: a description of the optimal synthesis is given and the loss functions are written in explicit form in terms of the coefficients $\{\lambda_i^j\}$. The results of this section are a reworking of the corresponding paper of the authors (1978*b*). In §5.3 we consider the same problem over a finite time horizon. This problem is considerably more complicated than in the finite horizon case since its solution is not stationary. Therefore, instead of explicit forms for the optimal synthesis and loss function we give an iterative construction of the optimal synthesis and describe some of its properties. The corresponding results were first published in the authors' work (1978*a*).

5.1 F-matrices and B-matrices

In §3.2, for the case of discrete time, we introduced the definitions of the hypothesis matrix classes and formulated Theorem 3.1, proved in the following sections. The main assertion of this theorem consists in the following. All hypothesis matrices $\{\lambda_i^j\}$ may be divided into two classes. For the matrices of one class (F-matrices) the loss function tends to a finite value as the length of the observation interval tends to infinity, and for the second class (B-matrices) the loss function tends to infinity. A similar theorem holds also for the case of continuous time and its proof essentially repeats the proof of Theorem 3.1. Therefore