Similarly, the existence of an optimal action rule for the problem of minimizing the functional (4.11) formulated in §4.2 may be established for the case of continuous time. For $\nu = \infty$, the analogue of Condition (A2) is

$$\lim_{s\to\infty} \inf_{\xi,\beta,\nu\geq s} E_\xi^\beta \int_s^\nu \xi Q(s)\beta^*(s)\,ds \geq 0.$$

However, to keep the proof free from unnecessary details, we will simply assume that $\phi^j(s) \geq 0$, $j = 1,\dots,m$, holds for the case $\nu = \infty$.

If we wish to obtain the existence of optimal action rules for $\nu = \infty$ for a sufficiently large class of functionals, for example, for all functionals of the form (4.11) with $\phi \geq 0$, then the topology on Π must have the following property. The weak convergence of the joint distribution of the values of $l_{1,k}$, $\tau_{1,k}$ for any $k = 1,2,\dots$ must follow from the convergence of the sequence of action rules $\beta^{(n)}$ in the given topology.

We introduce such a topology for Π in the case of a single hypothesis, i.e. for $N = 1$ (Theorem 4.1) and give an example which shows that for $N > 1$ such a topology cannot be introduced.

However, a topology which yields the compactness of the strategy space and the lower semicontinuity of the criterion functional may be introduced if we consider a richer set of randomized action rules. Since the description becomes overcomplicated in this case, we deal only briefly with this question and give without proof Theorem 4.3 stating the existence of an optimal action rule.

* * *

We study first the case of one hypothesis. The corresponding intensity vector is denoted by $\lambda := (\lambda^1,\dots,\lambda^m)$ and for reasons of simplicity we assume that $\lambda^j > 0$ for $j = 1,\dots,m$ and that the measure corresponding to the action rule β is given by P^β.

Consider the sample space of variables $\tau_1, l_1, \tau_2, l_2, \dots$, assuming that some τ coordinates may coincide, and some become $+\infty$ (in the case $\tau_r = +\infty$, then $\tau_{r+1} := +\infty$ and l_r is not defined and if $\tau_r < \infty$, then $\tau_{r+1} \geq \tau_r$). This space may be considered to be metric compact, if we consider it as a closed subset of a Tychonov product of the appropriate spaces (nonnegative half line with the point $+\infty$ attached and $\widehat{S}^m$). Functions corresponding to continuous functionals on the

compact obtained depend continuously on $\tau_{1,k}$ and $l_{1,k}$ for each fixed $k = 1,2,\dots$ and have a limit at infinity. The total set of all probability measures on this compact is a metric compact in the weak topology (see Dunford & Schwartz [1962]). Here the weak convergence of the sequence of measures P_n to the measure P is equivalent to the fact that for any $k = 1,2,\dots$ the sequence of joint distributions of the values $l_{1,k}, \tau_{1,k}$ induced by the measures P_n converge weakly to the joint distributions of these values induced by the measure P. Recall that the weak convergence of joint distribution functions is defined in terms of the corresponding Lebesgue–Stieltjes integrals of continuous functions with limits at infinity, here in $\tau_{1,k}$ since $l_{1,k}$ is discrete.

We introduce as a topology on the space of action rules the topology induced by the weak topology for measures on the given compact.

Theorem 4.1 *For the case of one hypothesis, the total set of action rules Π is a metric compact in the topology introduced above.*

Proof. According to the above discussion, if we can prove that the set of action rules is closed in the topology introduced, then it may be considered as a closed subset of a metric compact, and the theorem is proven.

First we will prove the following lemma.

Lemma 4.3 *On the product of a Borel space $\mathcal{X}$ and the closed half line $\{t : 0 \leq t \leq \infty\}$ (with attached point $+\infty$), suppose given a positive finite measure P. Then the following statements are equivalent:*

(a) For any measurable $B \subset \mathcal{X}$ and any t and Δ, $0 \leq t < \infty$, $0 < \Delta < \infty$, the inequalities

$$c_1(\Delta + o(\Delta))P\{B \times [t,\infty]\} \leq P\{B \times [t, t+\Delta)\} \leq c_2 \Delta P\{B \times [t,\infty]\} \qquad (4.30)$$

hold, where $0 \leq c_1 \leq c_2 < \infty$ and the function $o(\Delta)$ does not depend on B and t.

(b) There exists a measurable function $c(x,t)$ such that $c_1 \leq c(x,t) \leq c_2$ and for any B and any t, $0 \leq t < \infty$,

$$P\{B \times [0,t)\} = \int_B (1 - \exp\{-\int_0^t c(x,v)\,dv\})\mu(dx), \qquad (4.31)$$

where $\mu(\cdot)$ is a measure on $\mathcal{X}$ such that $\mu(B) := P\{B \times [0,\infty]\}$.

Proof. Statement (a) is obtained from (b) by substitution of expression (4.31) in the middle term of relation (4.30) and applying the inequality $c_1 \le c(x,t) \le c_2$ and the elementary inequalities $y - y^2/2 \le 1 - \exp\{-y\} \le y$.

We show that (b) follows from (a). Indeed, from the right-hand inequality of (4.30) it follows that $P\{B \times [t, t+\Delta)\} \le c_2 \Delta \mu(B)$, and this means that the measure P is absolutely continuous with respect to the product of the measure μ with Lebesgue measure on the half line on the set $\mathcal{X} \times [0,\infty)$. But then, by the Radon–Nykodym theorem there exists a measurable function $f(x,t)$ such that

$$P\{B \times [0,t)\} = \int_B \int_0^t f(x,v)\, dv \cdot \mu(dx), \tag{4.32}$$

for $0 \le f(x,t) \le c_2$ and $\int_0^\infty f(x,v)\, dv \le 1$. Accordingly, the right-hand inequality of (4.30) may be rewritten in the form

$$\int_B \int_t^{t+\Delta} [f(x,v) - c_2(1 - \int_0^v f(x,s)\, ds)] dv \cdot \mu(dx) \le 0,$$

or, taking account of the inequality $f(x,t) \le c_2$, in the form

$$\int_B \int_t^{t+\Delta} [f(x,v) - c_2(1 - \int_0^v f(x,s)\, ds)]\, dv \cdot \mu(dx)$$
$$\le c_2^2 \Delta \int_B \int_t^{t+\Delta} dv \cdot \mu(dx). \tag{4.33}$$

From (4.33) it follows that for any $\varepsilon > 0$ we can find a set of $dv \cdot \mu(dx)$ zero measure such that on its complement

$$f(x,v) - c_2(1 - \int_0^v f(x,s)\, ds) \le \varepsilon.$$

But from this it follows that a set of $dv \cdot \mu(dx)$ zero measure can be found such that on its complement

$$f(x,v) - c_2(1 - \int_0^v f(x,s)\, ds) \le 0. \tag{4.34}$$

Since the function $f(x,t)$ is defined up to sets of $dv \cdot \mu(dx)$ zero measure, then without loss of generality it may be assumed that (4.34) holds everywhere. Now set

$$c(x,v) := f(x,v)/(1 - \int_0^v f(x,s)\, ds).$$

It is easily checked that

$$\int_0^t f(x,v)\, dv = 1 - \exp\{-\int_0^t c(x,v)\, dv\}, \tag{4.35}$$

and this means that (4.31) holds by (4.32) and, according to (4.34) and the definition of $c(x,v)$, $c(x,v) \le c_2$. Now substitute (4.31) in the left-hand inequality of (4.30) and use the elementary inequality $1 - e^{-y} \le y$. Similarly to the above, we obtain that a set of $dv \cdot \mu(dx)$ zero measure may be found such that on its complement $c(x,v) \ge c_1$. On the exceptional set, set $c(x,v) := c_1$ and redefine the function $f(x,v)$ on this set of zero measure according to (4.35) by the formula $f(x,v) = c(x,v) \exp\{-\int_0^v c(x,s)\, ds\}$. Obviously, inequality (4.34) holds. Lemma 4.3 is proved. ■

To prove that the set Π is closed, it suffices to consider an arbitrary sequence of action rules β^n such that for any $k = 1, 2, \ldots$ the measures P^{β^n} corresponding to them on the sample space of values l_{1k}, τ_{1k} are weakly convergent to some measure and to show the existence of an action rule corresponding to this limit measure.

As was noted in §7 of the Appendix, corresponding to the action rule $\beta(t) := \{\beta_r(j_{1,r-1}, t_{1,r-1}, t),\ r = 1, 2, \ldots\}$ the distribution of the values l_{1r}, τ_{1r} has a density with respect to τ_{1r} which, for $0 = t_0 < t_1 < \cdots < t_r$, is given by

$$\Pi_{k=1}^r \left[\beta_k^{j_k}(j_{1,k-1}, t_{1k}) \lambda^{j_k} \exp\{-\int_{t_{k-1}}^{t_k} \lambda \beta_k^*(j_{1,k-1}, t_{1,k-1}, s)\, ds\}\right], \tag{4.36}$$

where $\lambda := (\lambda^1, \ldots, \lambda^m)$. Further, it is convenient to replace τ_{1r} and $\beta_r(\cdot)$ by $\tilde{\tau}_{1r}$ and $\tilde{\beta}_r(\cdot)$, where $\tilde{\tau}_k := \tau_k - \tau_{k-1}$, $k \ge 1$, $\tilde{\beta}(j_{1,r-1}, t_{1r}) := \beta_r(j_{1,r-1}, t_1, t_1 + t_2, \ldots, t_1 + \cdots + t_r)$, $r = 1, \ldots$. Moreover, when the number k and the index $j_{1,k-1}$ are fixed, we will write $\tilde{\beta}(\cdot)$ instead of $\tilde{\beta}_k(j_{1,k-1}, \cdot)$. For an arbitrary measurable subset B of the set $\{x = (t_1, \ldots, t_{k-1}) : t_i \ge 0\}$ define

$$P^{(n)}\{j, B \times [0,t)\} := P^{(\beta^n)}\{l_{1k} = j_1, \ldots, j_{k-1}, j, \tilde{\tau}_{1,k-1} \in B,\ \tilde{\tau}_k \in [0,t]\},$$
$$P^{(n)}\{B \times [0,t)\} := \sum_j P^{(n)}\{j, B \times [0,t)\},$$
$$P^{(n)}\{B\} := P^{(n)}\{B \times [0,\infty]\}.$$

From (4.36) we obtain

$$P^{(n)}\{j, B\times[0,t)\} = \int_B \int_0^t \phi^{(n)}(j,x,s)\,ds P^{(n)}(dx), \tag{4.37}$$

where

$$\phi^{(n)}\{j,x,s) = \lambda^j \tilde{\beta}^{(n)j}(x,s)\ \exp\{-\int_0^s \lambda \tilde{\beta}^{(n)*}(x,v)\,dv\}.$$

From (4.37) we may easily derive the following useful inequality:

$$P^{(n)}\{j, B\times[t,t+\Delta)\} \le \lambda^j \cdot \Delta \cdot P^{(n)}\{B\}. \tag{4.38}$$

Second, summing (4.37) with respect to j from 1 to m and using the equality $\int_0^t a(s)\ \exp\{-\int_0^s a(v)\,dv\}\,ds = 1 - \exp\{-\int_0^t a(v)dv\}$ we have that for the measure $P^{(n)}$ and the function $c^{(n)}(x,s) := \lambda\tilde{\beta}^{(n)*}(x,s)$, Condition (b) of Lemma 4.3 holds, which means that the inequalities

$$\begin{aligned}(\min_j \lambda^j)(\Delta + o(\Delta))P^{(n)}\{B\times[t,\infty)& \\ \le P^{(n)}\{B\times[t,t+\Delta)\}& \\ \le (\max_j \lambda^j)\cdot\Delta\cdot P^{(n)}\{B\times[t,\infty)\}&\end{aligned} \tag{4.39}$$

hold.

Finally, dividing (4.37) by λ^j and summing with respect to j from 1 to m we have

$$\begin{aligned}\sum_{j=1}^m (\lambda^j)^{-1} P^{(n)}\{j, B\times[0,t)\}& \\ = \int_B\int_0^t \exp\{-\int_0^s c^{(n)}(x,v)dv\}\,ds\,P^{(n)}(dx)& \\ = \int_0^t P^{(n)}\{B\times[s,\infty)\}\,ds.&\end{aligned} \tag{4.40}$$

The distributions $P^{(n)}\{j, B\times[0,t)\}$, $(P^{(n)}\{B\times[0,t)\})$ have, according to (4.37), uniformly bounded densities, thus if they are weakly convergent to some distribution then this distribution also has bounded density.

But if the limiting distribution has bounded density, then weak convergence is equivalent to convergence of the probability of any measurable set. Therefore, we may take limits with respect to n in the inequalities (4.38)–(4.40) and the limiting distribution P will also satisfy these inequalities. By the Radon–Nykodym theorem and by (4.38) for P it follows that there exist $f(j,x,s)$, such that $f(j,x,s) \le \lambda^j$ and

$$P\{j, B\times[0,t)\} = \int_B\int_0^t f(j,x,s)\,ds\,P(dx). \tag{4.41}$$

From Lemma 4.3 and (4.39) for P it follows that there exists $c(x,s)$, such that $\min_j \lambda^j \le c(x,s) \le \max_j \lambda^j$ and

$$P\{B\times[t,\infty)\} = \int_B \exp\{-\int_0^t c(x,s)\,ds\}P(dx). \tag{4.42}$$

From (4.41), (4.42) and the fact that (4.40) is also valid for P, it follows that we obtain for P

$$\sum_{j=1}^m (\lambda^j)^{-1} f(j,x,t) = \exp\{-\int_0^t c(x,s)\,ds\}. \tag{4.43}$$

Let

$$\tilde{\beta}^j(x,t) := (\lambda^j)^{-1} f(j,x,t)\exp\{\int_0^t c(x,s)\,ds\}. \tag{4.44}$$

From (4.43) it follows that $\tilde{\beta}_k(\cdot) = (\tilde{\beta}_k^1(\cdot),\ldots,\tilde{\beta}_k^m(\cdot)) \in S^m$. In (4.41), instead of $f(j,x,s)$ substitute the corresponding value from (4.44). Summing with respect to j from 1 to m and comparing with (4.42) we see that

$$\lambda\tilde{\beta}^*(x,t) = c(x,t).$$

Since the given inferences hold for any k, it follows that the density of limiting distribution P is represented by (4.36). This completes the proof of Theorem 4.1. ∎

Theorem 4.2 *In the case of one hypothesis for the problem of the minimization of the functional (4.11), there exists an optimal action rule β^ν for any $\nu < \infty$. If $\phi^j(s) \ge 0$, then an optimal action rule exists also for $\nu := \infty$, and $\lim_{\nu\to\infty} F_\nu = F_\infty$ and for $\nu \to \infty$ any limit point of β^ν will be the optimal action rule for the problem with $\nu = \infty$.*

Proof. As was said before, each function $\phi(t)$ is represented in terms of the sequence $\{\phi_r^j(j_{1,r-1}, t_{1,r-1}, t),\ r = 1, 2, \dots\}$, and

$$F_\nu^\beta := \sum_{r=1}^{\infty} E^\beta \phi_{\nu,r}^j(l_{1,r-1}, \tau_{1,r}),$$

where

$$\phi_{\nu,r}^j(j_{1,r-1}, t_{1,r-1}, t) := \begin{cases} 0 & \text{if } \ t > \nu \\ \phi_r^j(j_{1,r-1}, t_{1,r-1}, t) & \text{if } \ t < \nu. \end{cases}$$

Define

$$F_\nu^\beta(M, n) := \sum_{r=1}^{n} E^\beta \left[M \phi_{\nu,r}^{l_r}(l_{1,r-1}, \tau_{1,r}) \right].$$

By the assumption of the theorem and Remark 4.2, it may always be assumed that $\phi_{\nu r}^j(\cdot) \geq 0$ and this means that for fixed β the functions $F_\nu^\beta(M, n)$ are monotonically increasing with respect to n, M and ν. But the limit of monotonically increasing sequences and the integral of continuous functions are lower semicontinuous, so (see also Lemma 2.2) for proof of the theorem it suffices to show that, for any $\nu < \infty$ and $r = 1, 2, \dots$ and any positive bounded measurable function $f(j_{1,r-1}, t_{1,r})$, the function

$$E^\beta f(l_{1,r-1}, \tau_{1r}) \tag{4.45}$$

is continuous with respect to β. If the function $f(j_{1,r-1}, t_{1,r})$ is continuous with respect to $t_{1,r}$ for each fixed $j_{1,r-1}$, then expression (4.45) is continuous with respect to β by the definition of the topology on the space of action rules. It was shown in the proof of Theorem 4.1 that all joint distributions of the pairs $\tau_{1,r}, l_{1,r}$ have uniformly bounded densities with respect to $\tau_{1,r}$. Thus continuity with respect to β is preserved for all measurable bounded functions. Theorem 4.2 is proved. ■

* * *

We give an example of a sequence of action rules such that the corresponding sequence converges weakly, but for $N > 1$ the limiting measure does not correspond to any action rule.

Consider the case of two devices and let $\beta(t) := \beta^1(t)$. Let the sequence of action rules $\beta^{(n)}(t)$ be such that the corresponding $\beta_1^{(n)}(t_1)$ and $\beta_2^{(n)}(j, t_1, t)$ have the form $(j = 1, 2)$

$$\beta_1^{(n)}(t) = 1/2,$$

$$\beta_2^{(n)}(j, t_1, t) = \tilde{\beta}^{(n)}(t_1)$$

$$:= \begin{cases} a & \text{if } \ t_1 \in B_n = \bigcup_{r=0}^{\infty} \left\{ s : \dfrac{2r}{n} < s \leq \dfrac{2r+1}{n} \right\} \\ b & \text{if } \ t_1 \notin B_n, \end{cases}$$

$$0 < a,\ \ b < 1,\ \ a \neq b.$$

It is easy to see that for any hypothesis $P^{(n)}\{\tau_1 \in B_n\} \to 1/2$ and therefore, according to (4.36), for a hypothesis with parameters $\lambda = (\lambda^1, \lambda^2)$ we have

$$\begin{aligned} &\lim_{n\to\infty} P^{(n)}\{\tau_1 > t_1,\ l_1 = j,\ \tau_2 - \tau_1 > t\} \\ &\quad = \lim_{n\to\infty} \int_{t_1}^{\infty} \frac{\lambda^j}{2} \exp\{-(\lambda^1 + \lambda^2)s/2 \\ &\qquad\qquad - \int_0^t \left[(\lambda^1 - \lambda^2)\beta_2^{(n)}(j, s, s+v) + \lambda^2 \right] dv\}\, ds \\ &\quad = \lim_{n\to\infty} \int_{t_1}^{\infty} \frac{\lambda^j}{2} \exp\{-(\lambda^1 + \lambda^2)s/2 - \lambda^2 t(\lambda^1 - \lambda^2)\tilde{\beta}^{(n)}(s)t\}\, ds \\ &\quad = \int_{t_1}^{\infty} \frac{\lambda^j}{2} \exp\{-(\lambda^1 + \lambda^2)s/2\} f(t)\, ds, \end{aligned} \tag{4.46}$$

where

$$f(t) := \frac{1}{2} \exp\{-\lambda^2 t\}(\exp\{-(\lambda^1 - \lambda^2)at\} + \exp\{\lambda^1 - \lambda^2)bt\}). \tag{4.47}$$

Let the limiting measure be defined by some action rule to which $\beta_1(t)$ and $\beta_2(j, t_1, t)$ correspond. Then, comparing (4.46) and (4.36), we obtain that $\beta_1(t) \equiv 1/2$, $\beta_2(j, t_1, t_1 + t) = \tilde{\beta}(t)$ for $j = 1, 2$ and for any hypothesis

$$f(t) = \exp\{-\lambda^2 t\} \exp\{-\int_0^t (\lambda^1 - \lambda^2)\tilde{\beta}(s)\, ds\}.$$

From this and from (4.47) it follows directly that $\tilde{\beta}(s)$ depends on $(\lambda^1 - \lambda^2)$. If λ^1 and λ^2 are different for different hypotheses, this contradicts the definition of action rules.

Notice that the limiting measure corresponding to (4.46) is a convex linear combination (with the coefficients 1/2) of the measure corresponding to the action rules $\beta_1(t) \equiv 1/2$ and $\beta_2(j,t,s)$ with one of the measures identically equal to a and for the other equal to b. So our example also shows that the set of measures corresponding to action rules is *not convex*.

However, formula (4.46) allows a different interpretation. It can be said that $b_2(j,t_1,t)$ takes two values a and b, each with the probability 1/2. Such a *randomized* $\beta_2(j,t_1,t)$ addressed both hypotheses simultaneously.

In the general case, to give a *randomized action rule* the given sequence of functions $\{\beta_r(j_{1,r-1},t_{1r}),\ r=1,2,\ldots\}$ must be replaced by a given sequence of measures $\mu_r(\cdot|j_{1,r-1},t_{1,r-1})$ on the space of measurable functions $\alpha(t)$ taking values in S^m.[1] To introduce a topology on the space of such randomized action rules the problem formulated may be considered to be the discrete time problem of §2.2, where the control space $\mathcal{A}_n = \mathcal{A}$ consists of measurable functions $\alpha(t)$ taking values in S^m. The set $\mathcal{A}$ is considered as a subset of the space $L^\infty(0,\infty)$ with the weak topology induced by $L^1(0,\infty)$. It is known (see Dunford & Schwartz (1962, IV.13.6, IV.13.27)), that $\mathcal{A}$ is a metric compact in this topology, and that convergence is equivalent to convergence for each $t>0$ of the integral $\int_0^t \alpha(s)\,ds$. The state space $\mathcal{X}_n = \mathcal{X}$ corresponds to the time and value of the n^{th} jump of the process $X(t)$. An important rôle is played by the fact that the transition probabilities corresponding to the different hypotheses are absolutely continuous with respect to the transition probabilities corresponding, for example, to the case $\lambda := (1,\ldots,1)$.

The method of introduction of the L^∞ topology is described, for example, in Schäl [1979], where the statement of Theorem 4.2 is verified for functionals of a more general form than (4.11) which can depend on previous controls. By this method the existence of an optimal randomized action rule for each fixed $\xi \in S^N$ may be proved.

[1] The equivalent construction is given in terms of the continuous flow of σ-algebras of the observed process and control in Presman (1986).

The existence of an optimal *non*randomized action rule follows from the general statement concerning the Bayesian problem that for *any* strategy a not worse *nonrandomized strategy* can be found (see Dynkin & Yushkevitch 1976). As a result we have the following theorem.

Theorem 4.3 *In the minimization of functional (4.11) (for any ν, $0<\nu<\infty$) and any $\xi \in S^N$ there exists an optimal action rule. If $\phi^j(s) \geq 0$, then an optimal action rule exists also for $\nu = \infty$ and $\lim_{\nu\to\infty} F_\nu(\xi) = F_\infty(\xi)$.* ■

Since the above-mentioned approach is cumbersome and does not give an answer to the question of how to find an optimal strategy and what properties it possesses, we will not bother to prove Theorem 4.3, but will discuss in the sequel another reduction to discrete time after the transformation to a problem with complete information has already been effected. In this case, as usual, the simplest case is Markov. First we make a remark about properties of the function $F_\nu(\xi)$.

Remark 4.3 From the representation (4.4) for the measure P and the definition (4.25) of the functional $F_\nu(\xi)$, it follows that $F_\nu(\xi)$ for each fixed $\nu < \infty$ is a solution of a finite Bayesian problem, and this means that it satisfies the condition of Lemma 2.3. So, the statement of Theorem 2.2 holds for $F_\nu(\xi)$ in continuous time, and in particular $F_\nu(\xi)$ is convex with respect to ξ and continuous on the interior of the simplex S^N, and its projection on the interior of a face of any dimension is also continuous. ■

4.4 Reduction to a discrete time problem and existence of a Markov uniformly optimal strategy in the Markov case

At the end of §4.2, the initial basic scheme problem of minimization of the functional $F^\beta(\xi)$ with respect to all possible action rules $\beta \in \Pi$ (see (4.11)) was reduced to the problem of control with complete information of the processes $X(t)$, $\xi(t)$ with criterion functional (4.24), which can be rewritten as

$$F^\beta(\xi) = E_\xi^\beta \int_0^\infty \xi(s)Q(s)\beta^*(s)\,ds$$
$$= \sum_{n=1}^{\infty} E_\xi^\beta \int_{\tau_{n-1}}^{\tau_n} \xi(s)Q(s)\beta^*(s)\,ds, \tag{4.48}$$

where $Q(s) = \{q_i^j(s)\}$, $q_i^j(s) := \lambda_i^j\phi_i^j(s)$, $i = 1,\ldots,N$, $j = 1,\ldots,m$ and $\phi_i^j(s) := \{\phi_{in}^j(j_{1,n-1},t_{1,n-1},s),\ n = 1,\ldots\}$ are $I\!F_n$-predictable functions. We assume that if $\nu < \infty$, then $Q(s) := 0$ for $s > \nu$ and do not indicate the dependence of $F^\beta(\xi)$ and $Q(s)$ on ν. As before, suppose that $\phi_i^j(s) \geq 0$ for $\nu = \infty$.

Based on the sum representation of (4.48) and Lemma 4.2, this problem may be transformed to a problem with discrete time and complete information (see §§2.2 and 2.4), where the rôle of time is played by the *jump number*. This approach corresponds to consideration of the embedded chain for the process $X(t)$, $\xi(t)$, at jump moments t of this process. We thus consider the following problem in discrete time.

The state spaces $\mathcal{X}_n$, $n = 0,1,\ldots$ coincide with

$$\mathcal{X} = \{(j,t,\xi) : j = 1,\ldots,m,\ 0 \leq t < \infty,\ \xi \in S^N\} \cup \{\infty\}$$

equipped with the Borel σ-algebra and the control spaces coincide with $\mathcal{A}$, consisting of some element d_* and of all measurable functions defined on $[0,\infty)$ and taking values in S^m, so that $\mathcal{A}_* = \mathcal{A} \cup \{d_*\}$, where $\mathcal{A} := \{\alpha(\cdot) : \alpha(s) \in S^m,\ 0 \leq s < \infty\}$.

As in §4.3 the space $\mathcal{A}$ will be considered as a subspace of $L^\infty(0,\infty)$ with the weak topology with respect to $L^1(0,\infty)$.

The control d_* applies only in the state $\{\infty\}$, and as a result of its application the system stays in the same state. If at time k the system is in state (j,t,ξ) and the control $\alpha(\cdot)$ is applied, then the transition probabilities do *not* depend on k and are given by the formulae

$$P\{t_{k+1} < u|j,a\} = \begin{cases} 0 & \text{if } u < t \\ 1 - z(u|a) & \text{if } u \geq t, \end{cases} \tag{4.49}$$

$$P\{\xi_{k+1} = \Gamma^l(\xi(t_{k+1}|a)),\ j_{k+1} = l\ |j,a,t_{k+1},t_{k+1} < \infty\}$$
$$= \pi^l(t_{k+1}|a), \quad l = 1,\ldots,m,$$

where $a = (t,\xi,\alpha(\cdot))$, the functions $z(u|a)$, $\xi(u|a)$ and $\pi^l(u|\alpha)$ are defined in (4.28) and (4.29), and the transformations $\Gamma^l(\xi)$, $l = 1,\ldots,m$, coincide with transformations Γ^{1l} defined in (2.17), so that $(\Gamma^l\xi)_i = \lambda_i^l\xi_i/p^l(\xi)$. If $\lim_{u\to\infty} z(\nu|a) > 0$, then with probability equal to this limit the system jumps to state $\{\infty\}$ at time $k+1$.

Let the cost function at the n^{th} step $(n = 1,2,\ldots)$ be equal to 0 for the trajectory in state $\{\infty\}$. For the trajectory $j_{0,n-1}$, $t_{0,n-1}$, $\xi_{0,n-1}$, $\alpha_{1,n}(\cdot)$ the cost function is defined by the formula

$$q_n(j_{0,n-1},t_{0,n-1},\xi_{0,n-1},\alpha_{1,n}(\cdot))$$
$$:= \int_{t_{n-1}}^{\infty} \xi(s|a_n)Q_n(j_{1,n-1},t_{1,n-1},s)\alpha_n(s)z(s|a_n)\,ds. \tag{4.50}$$

Here $a_n := (t_{n-1},\xi_{n-1},\alpha_n(\cdot))$ and the matrix Q_n $(n = 1,2,\ldots)$ defines the representation of the $\mathcal{F}_t$-predictable matrix $Q(t)$ by a sequence of deterministic functions, so that

$$Q_n(\cdot) := \{q_{i,n}^j(\cdot),\ n = 1,2,\ldots\}, \quad q_{i,n}^j(\cdot) := \lambda_i^j\phi_{i,n}^j(\cdot).$$

Notice that in spite of the fact that all controls $\alpha(\cdot)$ are defined on the same interval $(0,\infty)$, the values of $\alpha(s)$ for $s < t$ do not play any rôle in the application of the control $\alpha(\cdot)$ in the state t,ξ.

Theorem 4.4 *The problem of minimizing the functional (4.48) with respect to all action rules $\beta \in \Pi$ is equivalent to the formulated discrete time minimization problem with initial point $j_0, t_0 := 0$, $\xi_0 := \xi$.*

Proof. Let a strategy π in the discrete time problem be put in correspondence with each action rule $\beta := \beta(s) := \{\beta_n(j_{1,n-1},t_{1,n-1},s),\ n = 1,2,\ldots\}$ in the following way. The strategy π puts a deterministic function $\alpha_n(s|k) := \beta_n(j_{1,n-1},\ t_{1,n-1},s)$ in correspondence with the trajectory $h = (j_{0,n-1},t_{0,n-1},\ \xi_{0,n-1},\alpha_{1,n-1}(\cdot))$, $n = 1,2,\ldots$. It is easily seen that these functions are the measurable images of $\mathcal{H}_{n-1}$ in $\mathcal{A}$, and this means that they actually define a strategy. We obtain, by the constructed correspondence and the choice of transition probabilities and a cost function for the problem with discrete time, that the cost of the strategy π for the initial state $j_0, t_0 := 0$, $\xi_0 := \xi$ coincides with $F^\beta(\xi)$ (see (4.48) and Lemma 4.2).