The first term of the right-hand side of (3.84) is nonnegative by the induction assumption and the fact that $f \geq 0$ implies $M_s f \geq 0$. The second term is nonnegative, since (see (3.75) and (3.76))

$$T_s W_{s+1}(\xi) \geq T W_{s+1}(\xi)$$
$$= \min\left[T^1 W_{s+1}(\xi), T^2 W_{s+1}(\xi)\right] = W_{s+2}(\xi),$$

which means that to prove (3.82) it is sufficient to check that $M^1 M^2 W_{s+1}(\xi) - M_s M^1 M^2 W_s(\xi) \geq 0$ for any $s = 1, 2, \ldots$. But to show this, it is sufficient to show that for $j = 1, 2$,

$$M^1 M^2 W_{s+1}(\xi) \geq M^j M^1 M^2 W_s(\xi). \tag{3.85}$$

By Lemma 3.1, the operators M^1 and M^2 are commutative, therefore to prove (3.85) it is sufficient to show that

$$W_{s+1}(\xi) \geq M^j W_s(\xi), \qquad j = 1, 2, \quad s = 1, 2, \ldots . \tag{3.86}$$

By Theorem 2.2, the function $W_s(\xi)$ is convex, which means that according to Property 2 of Lemma 3.1, $W_s(\xi) \geq M^j W_s(\xi)$. But for all s, from the nonnegativity of the cost functions, $W_{s+1}(\xi) \geq W_s(\xi)$. Therefore (3.86) is proven and hence (3.71) also for $\nu < \infty$.

From this, $\lim_{\nu\to\infty} W_\nu(\xi) = \lim_{\nu\to\infty} W'_\nu(\xi)$. But according to Theorem 3.1, for problems of the basic scheme $\lim W_\nu(\xi) = W_\infty(\xi)$. From the inequality $W'_\infty(\xi) \geq \lim_{\nu\to\infty} W'_\nu(\xi)$ (see (2.49) and (3.70)) it follows that (3.71) holds also for $\nu = \infty$. ∎

4 PROBLEM FORMULATION AND SOLUTION METHODS IN CONTINUOUS TIME

4.1 Reduction of continuous to discrete time

Chapter 2 gives a mathematical formulation of a situation which may be presented informally as follows. We have m devices, each of them generating independent Bernoulli (0 or 1) random variables with parameters depending on the number of devices. We have N hypotheses about parameter values and a given *a priori* distribution $\xi = (\xi_1, \ldots, \xi_N) \in S^N$ on the set of these hypotheses. At each moment of time only one of these devices may be used (i.e. the control action takes values in $\widehat{S}^m$) and the corresponding random variable may be observed. Accordingly, since the observations determine a payoff, it is required at each moment of time to decide which device should be used based on the data of which device was used at previous moments of time and the observations that were observed on each.

In the case of continuous time, the analogue of the *Bernoulli* process is the *Poisson* process. Therefore in continuous time it is natural to assume that under the i^{th} hypothesis the device with index j generates a Poisson process with parameter λ_i^j.

However, in continuous time it is not sufficient to take as controls functions taking values in $\widehat{S}^m$. A heuristic explanation of this fact was given in §1.6. Therefore, in continuous time the control $a(t)$ takes values in S^m, so that $a^j(t)$ corresponds to that *fraction* of a unit resource allocated to the j^{th} device at time t, and the *jump intensity* on this device under the i^{th} hypothesis equals $\lambda_i^j a^j(t)$.

In §4.2 a precise formulation of the continuous time problem based on the concepts of *martingale* and *point process* is given. The Appendix contains the facts necessary to understand this part. Moreover, in §4.2 it is shown that similarly to discrete time the problem may be formulated in the terms of control of the *a posteriori* probabilities of hypotheses.

Notice that in §4.2 only *nonrandomized* decisions are considered; these are given not by a measure, but by a function of the observations. In §4.3 it is proved that for the functionals considered the optimal values are achieved by members of this class.

In §4.4 it is shown that, by considering the given problem as a problem of sequential choice of controls on the intervals between jumps, it may be considered to be a control problem in discrete time. In this case the interpretation of randomized action rules becomes clear, as does the fact that, for the criterion functionals considered, randomization does not yield improvement in optimal values. With the reduction to a discrete time problem, a new definition of strategy is formulated (distinct from the action rule of §4.2) and the usual optimality equation is derived connecting the optimal values of functionals at the moments of successive jumps of the process.

Finally, in §4.5 a definition of *synthesis* of an optimal strategy is given and the *local optimality equation* is studied.

Some of the results presented in §§4.2–4.5 were initially published in Sonin (1976) and Presman & Sonin (1978*a*). The general question of control of jump processes similar in idea to the problems of this chapter is studied in Rishel (1970), Davis & Elliott (1977) and Yushkevitch (1980).

4.2 Basic scheme problem statement

Let $(\Omega, \mathcal{F})$ be a measurable space on which are given:

(a) a random vector $\theta = (\theta_1, \ldots, \theta_N)$ with values in $\widehat{S}^N$,

(b) an m-dimensional jump process $X(t) = \{X^1(t), \ldots, X^m(t)\}$, right continuous with $X(0) = 0$, such that $\Delta X(t)$ takes values either e_0^m or e_j^m, $j = 1, \ldots, N$.

These random variables are interpreted as follows: the set $\{\omega : \theta_i = 1\}$ corresponds to the i^{th} hypothesis being *true*; $X^j(t)$ represents the *number of events* observed on the j^{th} device up to time t.

It is equivalent to giving the process $X(t)$ to give the sequence $\{\tau_n(\omega), l_n(\omega)\}$, where $\tau_n = \tau_n(\omega)$ is the *time* of the n^{th} jump of the process $X(t)$ and $l_n := l_n(\omega)$ indicates the coordinate in which the jump at time τ_n took place. Here $\tau_1 > 0$, $\tau_{n+1} > \tau_n$ for $\tau_n < \infty$, $\tau_{n+1} := \infty$ for $\tau_n = \infty$, $n = 1, 2, \ldots$, and l_n is defined only for $\tau_n < \infty$. The process $X(t)$ is an *m-variate counting process*, and $\{\tau_n, l_n\}$ is an *m-variate point process* (see §2 of the Appendix).

Let $\mathcal{F}_t := \sigma(X(s),\ 0 \le s \le t)$ denote the σ-algebra generated by the values of the process $X(s)$ up to time t and set $\mathcal{F}_t^\theta := \sigma(\theta) \vee \mathcal{F}_t$, $\mathbb{F} := \{\mathcal{F}_t\}_{t \ge 0}$ and $\mathbb{F}^\theta := \{\mathcal{F}_t^\theta\}_{t \ge \theta}$. Moreover, let

$$\tau_{1,n} := (\tau_1, \ldots, \tau_n), \quad l_{1,n} := (l_1, \ldots, l_n), \quad \mathcal{F}_{\tau_n} := \sigma(\tau_{1,n}, l_{1,n}). \tag{4.1}$$

An *action rule* is an *$\mathbb{F}$-predictable* function $\beta(t) := \{\beta^1(t), \ldots, \beta^m(t)\}$ taking values in S^m. The coordinate $\beta^j(t)$ is understood as the *fraction* of a unit *resource* directed to the j^{th} device at time t, which depends on the results of the observations up to time t.

Let a *hypothesis matrix* $\Lambda := \{\lambda_i^j\}$ $(i = 1, \ldots, N,\ j = 1, \ldots, m)$ be given, where $0 \le \lambda_i^j < \infty$ and λ_i^j defines the *intensity* of the Poisson process corresponding to the j^{th} device under the i^{th} hypothesis in the case in which all the unit resource is directed at that device.

If the space $(\Theta, \mathcal{F})$ is sufficiently rich, then, as follows from §6 of the Appendix, the values θ and X may be given in terms of ω in such a way that to each $\xi \in S^N$ (defining an *a priori* distribution on the set of hypotheses) and to each action rule $\beta = \beta(t)$ there corresponds a measure P_ξ^β on $\mathcal{F}_\infty^\theta$ such that

$$P_\xi^\beta\{\theta_i = 1\} = \xi_i, \tag{4.2}$$

and the process coordinates

$$X(t) - \int_0^t \theta\Lambda \ \mathrm{diag}\ \beta(s)\, ds \tag{4.3}$$

are orthogonal *$\mathbb{F}^\theta$-martingales* (the integral in (4.3) is the *compensator* of the process $X(t)$).

From (4.2) and (4.3) it follows that for the measure P_ξ^β we have the representation

$$P_\xi^\beta = \sum_{i=1}^N \xi_i P_{e_i}^\beta := \sum_{i=1}^N \xi_i P_i^\beta, \tag{4.4}$$

and the measure P_i^β is concentrated on the set $\{\theta_i = 1\}$. This implies that the elements of the matrix $\theta^* X(t) - (\text{diag}\,\theta)\Lambda \int_0^t \text{diag}\,\beta(s)\,ds$ having the form

$$\theta_i(X^j(t) - \int_0^t \lambda_i^j \beta^j(s)\,ds), \qquad i = 1,\dots,N, \quad j = 1,\dots,m, \qquad (4.5)$$

are also orthogonal $I\!F^\theta$-martingales. Obviously, the expressions in the brackets in (4.5) are also orthogonal $I\!F$-martingales with respect to the measure P_i. The value $\lambda_i^j\beta^j(s)$ is naturally called the *local intensity* of the process $X^j(t)$ *under the* i^{th} *hypothesis*, since an exact meaning can be given to the equality

$$P(t \le \tau_{n+1} < t + dt, l_{n+1} = j | \mathcal{F}_{\tau_n}, \tau_{n+1} \ge t, \theta_i = 1) I\{\tau_n < t < \tau_{n+1}\}$$
$$= \theta_i \lambda_i^j \beta^j(t) I\{\tau_n < t \le \tau_{n+1}\}\,dt. \qquad (4.6)$$

In particular, if $\beta^j(t,\omega) \equiv c$ on some time interval, then on this interval under the i^{th} hypothesis the process $X(t)$ will be a Poisson process with intensity $\lambda_i^j c$.

Remark 4.1 Note that the measure P_ξ^β is defined uniquely by the integral in (4.3). Therefore, the consideration of the natural class of *$I\!F$-adapted* action rules does not give anything new, because for any $I\!F$-adapted function $\tilde\beta := \tilde\beta(t)$ we can find an action rule given by an $I\!F$-predictable function such that the integrals in (4.3) coincide. ■

Further, we will not distinguish between action rules which correspond to the same measure P_ξ^β. The set of *all* possible action rules is denoted by Π. Since we will *assume* that the intensity of jumps is bounded, then, according to §6 of the Appendix, only a finite number of jumps can occur on each finite time interval with probability 1.

As in discrete time, for $\nu < \infty$ the functional $V_\nu^\beta(\xi)$ denotes the expected number of jumps (successes) up to time ν using action rule β and with *a priori* distribution ξ of the hypotheses. Since (4.3) is martingale,

$$V_\nu^\beta(\xi) := E_\xi^\beta \sum_{j=1}^m X^j(\nu) = E_\xi^\beta \int_0^\nu \theta\Lambda\beta^*(s)\,ds, \qquad (4.7)$$

where E_ξ^β denotes expectation with respect to the measure P_ξ^β.

Consider the problem of maximization of the functional (4.7). Accordingly, let

$$V_\nu(\xi) := \sup_{\beta\in\Pi} V_\nu^\beta(\xi). \qquad (4.8)$$

As in the case of discrete time, it is convenient to replace the maximization of the functional (4.7) by the minimization of *loss*, which up to time ν is given by

$$W_\nu^\beta(\xi) = \nu \sum_{i=1}^N \xi_i\lambda_i - V_\nu^\beta(\xi) = E_\xi^\beta \int_0^\nu \theta(\tilde\Lambda - \Lambda)\beta^*(s)\,ds, \qquad (4.9)$$

where λ_i and $\tilde\Lambda$ are defined similarly to the definitions in §2.1 (see (2.7)). By the nonnegativity of the function under the integral sign in (4.7), the loss $W_\nu^\beta(\xi)$ may also be considered for $\nu = \infty$. Denote by

$$W_\nu(\xi) = \inf_{\beta\in\Pi} W_\nu^\beta(\xi) \qquad (4.10)$$

the (optimal) loss function up to time $\nu \le \infty$.

Basically we will be interested in the problem of maximization of the expected number of successes, or, which is the same, the minimization of loss. However, some of our results will hold for the problem of minimization of an arbitrary functional in the form:

$$F_\nu^\beta(\xi) = E_\xi^\beta \int_0^\nu \phi(s)\,dX^*(s), \qquad (4.11)$$

where $\phi(t)$ is an m vector-valued measurable $I\!F$-predictable function bounded from below, $\nu \le \infty$. The case $\nu = \infty$ is considered under the assumption that the integral and the expectation in (4.11) are defined for any β. When it is clear which ν is under consideration the notation will be simplified and we will simply write $F^\beta(\xi)$.

We show that there exist two other convenient representations (see (4.12) and (4.13)) for the functional (4.11).

If $\Omega = \Theta \times \Omega'$, then any m-measurable $I\!F^\theta$-predictable function $\phi(t)$ may be represented as $\phi(t) = \theta\Phi(t)$, where the elements of the matrix $\Phi(t)$, $i = 1,\dots,N$, $j = 1,\dots,m$, are $I\!F$-predictable functions. Since (4.11) does not depend on the structure of the space Ω, then without loss of generality we will consider $\phi(t) = \theta\Phi(t)$. Using the fact that (4.3) is a martingale, we have

$$F_\nu^\beta(\xi) = E_\xi^\beta \int_0^\nu \theta\Phi(s)(\operatorname{diag}\beta(s))\Lambda^*\theta^*\,ds = E_\xi^\beta \int_0^\nu \theta Q(s)\beta^*(s)\,ds, \tag{4.12}$$

where $Q(s) := \{q_i^j(s)\}$, $q_i^j(s) := \lambda_i^j\phi_i^j(s)$, $i = 1,\ldots,N$, $j = 1,\ldots,m$, and the last equality in (4.12) follows in an elementary way taking account of the equations $\theta_i\theta_i = \theta_i$, $\theta_i\theta_j = 0$ for $i \neq j$.

According to (4.7) and (4.9), $Q(s) = \Lambda$ corresponds to the number of successes, i.e. $\phi_i^j = 1$ and $Q(s) = \bar\Lambda - \Lambda$ corresponds to loss, i.e. $\phi_i^j(s) = \lambda_i/\lambda_i^j - 1$ (if $\lambda_i^j = 0$, assume $\phi_i^j = 0$).

Remark 4.2 Let I denote the matrix of size $N\times m$, with all its elements equal to 1. Then, obviously, $\int_0^\nu \theta I\beta^*(s)\,ds = \nu$. If $\nu < \infty$, then minimization of the functional $F_\nu^\beta(\xi)$ is equivalent to minimization of $F_\nu^\beta(\xi) + c\nu = E_\xi^\beta \int_0^\nu \theta(Q(s) + cI)\beta^*\,ds$. From this and from the boundedness from below of the function $\phi(t)$, it follows that for $\nu < \infty$ the function ϕ may always be considered to be nonnegative. ■

Another representation of the functional (4.11) is connected with the representation of any $\mathbb{F}$-predictable function $f(t)$ by a sequence of deterministic Borel functions $\{f_n(j_{1,n-1}, t_{1n}),\ n = 1,2,\ldots\}$, where $t_{1n} := (t_1,\ldots,t_n)$, $j_{1n} := (j_1,\ldots,j_n)$, $0 < t_1 < \cdots < t_n < \infty$, $j_k = 1,\ldots,m$ for $k = 1,\ldots,n$ (see §3 of the Appendix). Namely, for $n \geq 1$,

$$f(t,\omega) = \sum_{n=1}^\infty f_n(l_{1,n-1}(\omega), \tau_{1,n-1}(\omega), t) I\{\tau_{n-1}(\omega) < t \leq \tau_n(\omega)\}.$$

Sometimes we will simply write $f(t) = \{f_n(j_{1,n-1}, t_{1,n-1}, t), n = 1,2,\ldots\}$ to indicate this representation.

Correspondingly, let

$$\phi_i^j(t) := \{\phi_{in}^j(j_{1,n-1}, t_{1,n-1}, t),\ n = 1,2,\ldots\},$$

assuming that $\phi_i^j(t) \equiv 0$ for $t > \nu$, and rewrite (4.11) in the form

$$F_\nu^\beta(\xi) = \sum_{n=1}^\infty E_\xi^\beta \phi^{l_n}(\tau_n) = \sum_{n=1}^\infty E_\xi^\beta \phi_n(\theta, l_{1n}, \tau_{1n}), \tag{4.13}$$

where $\phi_{in}^{j_n}(j_{1,n-1}, t_{1n}) := \phi_n(e_i, j_{1n}, t_{1n})$. Since any sequence of Borel functions $\{f_n(j_{1,n-1}, t_{1,n}),\ n = 1,\ldots\}$ defines an $\mathbb{F}$-predictable function, then any functional having the form of the right-hand side of (4.13) can be rewritten in the form (4.11).

* * *

We now introduce the *a posteriori probability of hypotheses* vector

$$\xi^\beta(t) := E_\xi^\beta(\theta|\mathcal{F}_t). \tag{4.14}$$

In discrete time the process $\xi^\beta(n)$ may be chosen independently of the action rule β. In the case of continuous time this cannot be done. This is connected with the fact that the filtration of σ-algebras $\{\mathcal{F}_n\}$ in discrete time is generated by observations and controls, but in continuous time the filtration $\mathbb{F}$ is generated only by the observations. We will omit the index β for $\xi^\beta(t)$ when this will not lead to confusion.

For the process $X(t)$ the corresponding filtration $\mathbb{F}$ of σ-algebras is *right continuous* (see §4 of the Appendix). Therefore the process $\xi(t)$ may be assumed to be right continuous with left-hand limits.

As in discrete time, for $\xi \in S^N$ we set $p(\xi) := \xi\Lambda$, so that

$$p(\xi) = \{p^1(\xi),\ldots,p^N(\xi)\}, \quad p^j(\xi) = \sum_{i=1}^N \lambda_i^j\xi_i. \tag{4.15}$$

Take the conditional expectation with respect to $\mathcal{F}_t$ of the martingale (4.3). Noting that $\mathcal{F}_t \subset \mathcal{F}_t^\theta$ and taking the conditional expectation with respect to $\mathcal{F}_{s-}$ under the integral sign, we see that the process

$$Y(t) = X(t) - \int_0^t p(\xi(s-))\operatorname{diag}\beta(s)\,ds \tag{4.16}$$

is an $\mathbb{F}$-martingale, and this means that the integral on the right-hand side of (4.16) coincides with the compensator of the process $X(t)$ with respect to the filtration $\mathbb{F}$. The matrix characteristic of $Y(t)$ equals, according to §6 of the Appendix,

$$< Y^*(t), Y(t) > = \int_0^t \operatorname{diag} p(\xi(s-))\operatorname{diag}\beta(s)\,ds. \tag{4.17}$$

Now we describe how the *a posteriori* probabilities are changing. Unlike the case in discrete time, when $\xi(n)$ changes by jumps at each moment of time, now $\xi(t)$ has a jump at the jump moments of the process $X(t)$ and in the intervals between jumps the trajectories of the process $\xi(t)$ are the solutions of ordinary differential equations. This can be written generally as the following stochastic integral equation (4.18), where the integrals may be understood here as Lebesgue–Stieltjes integrals for each ω.

Lemma 4.1 *For P_ξ^β almost every ω*

$$\xi^*(t) = \xi^* + \int_0^t g(\xi(s_-))\,[\text{diag}\; p(\xi(s_-))]^{-1}\, dY^*(s), \tag{4.18}$$

where $g(\xi) := (\text{diag}\;\xi)\Lambda - \xi^* p(\xi)$ and it is assumed in the definition of the inverse matrix $[\text{diag}\; p(\xi)]^{-1}$ that if $p^j(\xi) = 0$, then $1/p^j(\xi) := 0$.

Proof. Consider the process $v^*(t)$ given by the right-hand side of formula (4.18). According to §8 of the Appendix, v^* is a square integrable martingale. From (4.16) and (4.17) it follows that for any square integrable martingale of the form

$$\zeta(t) = \int_0^t (dY(s)) \cdot f(s), \tag{4.19}$$

where $f(s)$ is a matrix with predictable elements,

$$E_\xi^\beta v^*(t)\zeta(t) = E_\xi^\beta \int_0^t g(\xi(s))\,[\text{diag}\;\beta(s)]\, f(s)\, ds \tag{4.20}$$

holds (see §8 of the Appendix).

On the other hand, from the definition of the process $\xi(t)$ and (4.16) we have

$$\begin{aligned} &E_\xi^\beta \xi^*(t)\zeta(t) \\ &\quad = E_\xi^\beta \theta^* \zeta(t) = E_\xi^\beta \theta^* \int_0^t dY(s) f(s) \\ &\quad = E_\xi^\beta \int_0^t (\theta^*\, dX(s) - (\text{diag}\;\theta)\Lambda(\text{diag}\;\beta(s))\, ds) f(s) \\ &\qquad + E_\xi^\beta \int_0^t \left[(\text{diag}\;\theta)\Lambda - \theta^* p(\xi(s))\right] \text{diag}\;\beta(s) f(s)\, ds. \end{aligned} \tag{4.21}$$

Since the elements of the matrix $f(s)$ are $I\!F$-predictable functions, and the elements of the matrix $\theta^* X(s) - (\text{diag}\;\theta)\Lambda \int_0^s \text{diag}\;\beta(s)\, ds$

are $I\!F$-martingales (see (4.5)), the first term in the right-hand side of (4.21) is equal to 0. Taking the conditional expectation with respect to $\mathcal{F}_s$ inside the integral in the second term we obtain

$$E_\xi^\beta \xi^*(t)\zeta(t) = E_\xi^\beta \int_0^t g(\xi(s))(\text{diag}\;\beta(s)) f(s)\, ds. \tag{4.22}$$

In the right-hand side of (4.22), $\xi(s)$ may be replaced by $\xi(s_-)$ and therefore comparing (4.20) and (4.22) we see that

$$E_\xi^\beta \left[(\xi^*(t) - v^*(t))\zeta(t)\right] = 0. \tag{4.23}$$

Since $\xi^*(t) - v^*(t)$ is an $I\!F$-martingale and, according to §8 of the Appendix, an arbitrary martingale can be represented as in (4.19), then as $\zeta(t)$ in (4.19) we may take $\xi(t) - v(t)$. Therefore the equality $\xi(t) = v(t)$ holds P_ξ^β almost everywhere for each fixed t. But $\xi(t)$ and $v(t)$ are continuous from the right, and this means immediately that the equality $\xi(t) = v(t)$ holds P_ξ^β almost surely for all t. The proof is complete. ■

By (4.14) the functional $F_\nu^\beta(\xi)$ (see (4.11) and (4.12)) can be rewritten as

$$F_\nu^\beta(\xi) = E_\xi^\beta \int_0^\nu \xi(s) Q(s) \beta^*(s)\, ds. \tag{4.24}$$

We want to find

$$F_\nu(\xi) := \inf_{\beta\in\Pi} F_\nu^\beta(\xi). \tag{4.25}$$

In the initial formulation a fixed ξ and β define a measure on $\mathcal{F}_\infty^\theta$. As is shown by (4.24), in solving the minimization problem for the functional (4.11) with respect to all possible action rules it suffices to consider measures on $\mathcal{F}_\infty$. Similarly to the discrete case, the introduction of *a posteriori* probabilities allows for fixed ξ and β a measure to be relatively simply given on $\mathcal{F}_\infty$ and the random variable θ may be excluded from consideration.

Construction of such a measure and the transformation of the functional to (4.24) corresponds to transformation from the initial control problem for the random process $X(t)$ with incomplete information (i.e. with unknown value θ) to the control of a pair $X(t)$, $\xi(t)$ of processes, where $\xi(t)$ satisfies the stochastic integral equation (4.18).

According to §7 of the Appendix, to construct the measure on $\mathcal{F}_\infty$ it suffices to define the conditional distributions of the time and coordinate index of the next jump by the formulae

$$P\{\tau_{n+1}-\tau_n > s|\mathcal{F}_{\tau_n}\} = \exp(-\int_{\tau_n}^{\tau_n+s}\beta(v)p^*(\xi(v))\,dv) \quad \text{for} \quad \tau_n<\infty, \tag{4.26}$$

$$P\{l_{n+1}=j|\mathcal{F}_{\tau_n}\wedge\sigma(\tau_{n+1})\} = \frac{\beta^j(\tau_{n+1})p^j(\xi(\tau_{n+1}-))}{\beta(\tau_{n+1})p^*(\xi(\tau_{n+1}-))} \quad \text{for} \quad \tau_{n+1}<\infty. \tag{4.27}$$

The right-hand side in (4.26) is conveniently expressed in terms of τ_{1n} and l_{1n}. As mentioned before, the $I\!\!F$-predictable function $\beta(t)$ is represented in terms of a sequence of deterministic functions $\{\beta_n(j_{1,n-1},t_{1,n}),\ n=1,2,\ldots\}$. The function $\beta_n(j_{1,n-1},t_{1,n-1},t)$, considered as a function of t, we will call the *control* between the $(n-1)^{\text{st}}$ jump and the n^{th} jump, and an *action rule* we will understand to be both an $I\!\!F$-predictable function $\beta(t)$ taking values in S^m and the sequence $\{\beta_n(j_{1,n-1},t_{1,n-1},t)\}$ of controls between the jumps.

To express the right-hand side of (4.26) in terms of $\tau_{1,n}$ and $l_{1,n}$, $\beta(s)$ must be replaced by $\beta_{n+1}(l_{1,n},\tau_{1,n},s)$. The function $\xi(s-)$ of (4.27) on the interval $(\tau_n,\tau_{n+1}]$ is also very simply expressed in terms of $\xi(\tau_n)$ and $\beta_{n+1}(l_{1,n},\ \tau_{1,n},s)$ using equation (4.18).

We will introduce the following notation. Let $a:=(t,\xi,\alpha(\cdot))$, where $\alpha(s):=(\alpha^1(s),\ldots,\alpha^m(s))$ is a measurable vector-valued function defined for $s>0$ and taking values in S^m. Denote by $\xi(s|a)$ and $z(s|a)$ the solutions for $s\geq t$ of the differential equations (compare (1.10))

$$\begin{aligned}\frac{d}{ds}\xi^*(s|a) &= -g(\xi(s|a))\alpha^*(s)\\ &= [\xi^*(s|a)\xi(s|a) - \operatorname{diag}\xi(s|a)]\Lambda\alpha^*(s)\\ \frac{d}{ds}z(s|a) &= -z(s|a)\alpha(s)p^*(\xi(s|a))\\ &= -z(s|a)\xi(s|a)\Lambda\alpha^*(s),\end{aligned} \tag{4.28}$$

with initial conditions $\xi(t|a)=\xi$, $z(t|a)=1$, and let

$$\pi^j(s|a) := \alpha^j(s)p^j(\xi(s|a))/\alpha(s)p^*(\xi(s|a)). \tag{4.29}$$

Lemma 4.2 *Let* $\beta=\{\beta_n(j_{1,n-1},t_{1,n-1},t),\ n=1,\ldots\}$ *be an action rule and* $\bar{a}:=\{\tau_n,\xi(\tau_n),\beta_{n+1}(l_{1n},\tau_{1n},t)\}$. *Then:*

(1) $P\{\tau_{n+1}-\tau_n<s|\mathcal{F}_{\tau_n}\}=1-z(\tau_n+s|\bar{a})$ *for* $\tau_n<\infty$.

(2) $P\{l_{n+1}=j|\mathcal{F}_{\tau_n}\vee\sigma(\tau_{n+1})\}=\pi^j(\tau_{n+1}|\bar{a})$ *for* $\tau_{n+1}<\infty$.

(3) $\xi(s_-)\,I\{\tau_n<s\leq\tau_{n+1}\}=\xi(s|\bar{a})\,I\{\tau_n<s\leq\tau_{n+1}\}$.

(4) If $l_{n+1}=j$, *then* $\xi(\tau_{n+1})=\Gamma^{j_1}\xi(\tau_{n+1}|a)$ *for* $\tau_{n+1}<\infty$, *where the transformation* Γ^{1j} *is defined in (2.17).*

(5) For any predictable function $f(t)=\{f_{n+1}(j_{1n},t_{1n},t),n=0,1,\ldots\}$

$$E_\xi^\beta\left\{\int_{\nu\wedge\tau_n}^{\nu\wedge\tau_{n+1}} f(s)\,ds|\mathcal{F}_{\tau_n}\right\} = \int_{\nu\wedge\tau_n}^{\nu} f_{n+1}(l_{1,n},\tau_{1,n},s)z(s|\bar{a})\,ds.$$

Proof. Statements 3 and 4 follow directly from the stochastic integral equation (4.18), written in the form of differential equations between the jumps and of a relation between pre- and post-jump states at the jump times. Statements 1, 2 and 5 are obtained from (4.26) by substituting the function $\beta_{n+1}(l_{1,n},\tau_{1,n},s)$ in place of $\beta(s)$, replacing $\xi(s_-)$ by the corresponding expressions from Statement 3, and replacing $f(s)$ by the function $f_{n+1}(l_{1,n},\tau_{1,n},s)$.

4.3 Existence of an optimal action rule

As mentioned in §2.3, to prove the existence of the optimal action rule in discrete time for $\nu<\infty$, it suffices to introduce a topology on the space of action rules Π, so that Π is compact in this topology, and the functional $W_\nu^\pi(\theta)$ is lower semicontinuous with respect to π. If, moreover, Condition (A2) holds, then semicontinuity follows and this means the existence of an optimal action rule for $\nu=\infty$ and the optimality of the pointwise limit (as $\nu\to\infty$) of the finite horizon optimal action rules.