$\{e_j^k, j = 1, \ldots, k\}$, where e_j^k is the k-dimensional vector with the j^{th} coordinate equal to 1 and the rest equal to 0, and finally $\widehat{S}_0^k := \widehat{S}^k \cup e_0^k$, where e_0^k is the k-dimensional vector with all coordinates 0. Sometimes, if it is clear which dimension k we are discussing, we will write e_j, $j = 0, 1, \ldots, k$.

Let $(\Omega, \mathcal{F})$ be a measurable space on which are given

(a) a random vector $\theta = (\theta_1, \ldots, \theta_N)$ with values in $\widehat{S}^N$,

(b) a sequence of random vectors $a := \{a(n),\ n = 1, 2, \ldots\}$, where $a(n) := (a^1(n), \ldots, a^m(n))$ takes values in $\widehat{S}^m$,

(c) a sequence of m-dimensional random vectors $X := \{X(n), n = 0, 1, \ldots\}$, where $X(n) := (X^1(n), \ldots, X^m(n))$, $X(0) := e_0$, and $\Delta X(n) := X(n) - X(n-1)$ takes values in $\widehat{S}_0^m$; in this case if $a(n) := e_j$, then $\Delta X(n)$ equals either e_0 or e_j.

These values are interpreted in the following way. The set $\{\omega : \theta_i = 1\}$ corresponds to the situation in which the i^{th} *hypothesis* holds. The coordinate of $a(n)$ equal to 1 indicates the *device number* observed at time n. The value of the process X at each moment either remains as it was previously or one of its coordinates increases by 1, so $X^j(n)$ corresponds to the number of jumps (successes) observed on the j^{th} device up to time n inclusive.

Denote by $h(n) = (a(1), \Delta X(1), \ldots, a(n), \Delta X(n))$ the *history* of the observable components of the process up to time n, $1 \le n \le \infty$, $h(0) := X(0)$, and introduce for $n \ge 0$ the following σ-algebras:

$$\begin{aligned} \mathcal{F}_n &:= \sigma\{h(n)\}, & \mathcal{F}_n^\theta &:= \mathcal{F}_n \vee \sigma\{\theta\}, \\ \widetilde{\mathcal{F}}_{n+1} &:= \mathcal{F}_n \vee \sigma\{a(n+1)\}, & \widetilde{\mathcal{F}}_{n+1}^\theta &:= \widetilde{\mathcal{F}}_{n+1} \vee \sigma\{\theta\}. \end{aligned} \tag{2.1}$$

Here, the σ-algebra $\mathcal{F}_n$ is interpreted as the class of *events* defined by which devices were observed up to time n inclusive and which values of the process X were observed on them. The definition of $\mathcal{F}_n^\theta$ implies that in addition to the information given by $\mathcal{F}_n$ the "true" hypothesis is known. $\widetilde{\mathcal{F}}_{n+1}$ and $\widetilde{\mathcal{F}}_{n+1}^\theta$ differ from $\mathcal{F}_n$ and $\mathcal{F}_n^\theta$ respectively by the fact that it is also known which device is observed at time $n+1$.

A *decision* as to what device to observe (which device to use) at time n (such a decision may also be randomized) is taken on the basis of which devices were observed up to time $n-1$ inclusive and which

results were obtained on them. In relation to these decisions, we will term an *action rule* a sequence $\beta := \{\beta(n),\ n = 1, 2, \ldots\}$, where

$$\beta(n) := (\beta^1(n), \ldots, \beta^m(n))$$

takes values in S^m and is $\mathcal{F}_{n-1}$ measurable for each fixed n. Here $\beta^j(n)$ defines the probability with which the j^{th} device should be observed at time n.[1]

We suppose given a *hypothesis matrix* $\Lambda = \{\lambda_i^j\}$, $0 \le \lambda_i^j \le 1$, $i = 1, \ldots, N$, $j = 1, \ldots, m$ (the index i corresponds to the i^{th} row of the matrix). Here the number λ_i^j defines the probability that under the i^{th} hypothesis a jump will be observed on the j^{th} device on condition that exactly this device is used.

If the space $(\Omega, \mathcal{F})$ is sufficiently rich, then it may be assumed that to each $\xi = (\xi^1, \ldots, \xi^N) \in S^N$ (defining an *a priori* distribution on the set of hypotheses) and to each action rule β corresponds a measure P_ξ^β on $\mathcal{F}_\infty^\theta = \bigvee_{n=1}^\infty \mathcal{F}_n^\theta$ such that for $i = 1, \ldots, N$, $j = 1, \ldots, m$, $n \ge 1$, the following hold:

$$P_\xi^\beta\{\theta_i = 1\} = \xi_i \tag{2.2}$$

$$P_\xi^\beta\{a^j(n) = 1 | \mathcal{F}_{n-1}^\theta\} = \beta^j(n) \tag{2.3}$$

$$P_\xi^\beta\{X^j(n) - X^j(n-1) = 1 | \widetilde{\mathcal{F}}_n^\theta\} = (\sum_{i=1}^N \theta_i \lambda_i^j) a^j(n)\,. \tag{2.4}$$

Actually, Ω can be taken, as indicated above, as the *sample* (*path*) *space* of values of X, a, θ. Since the σ-algebra $\mathcal{F}_n^\theta$ for fixed n consists of a finite number of events, the measure P_ξ^β for each fixed n is defined on $\mathcal{F}_n^\theta$ by (2.2)–(2.4) recursively and by Kolmogorov's theorem can therefore be uniquely extended to $\mathcal{F}_\infty^\theta$.

The relation (2.4) means that if the j^{th} device ($a^j(n) := 1$) is observed at time n, then under the i^{th} hypothesis ($\theta_i := 1$) the probability of realizing a 1 (a jump) equals λ_i^j and the jump can only occur on the device which is observed.

[1] It will be explained in §2.4 why instead of the usual terminology the term "strategy" is replaced by the term "action rule."

Remark 2.1 Relation (2.4) is equivalent to the fact that the coordinates of the process

$$X(n) - \theta\Lambda \sum_{s=1}^{n} \operatorname{diag}\ a(s)$$

are $\tilde{\mathcal{F}}^{\theta}_{n+1}$ *martingales.* According to (b) and (c) above,

$$a^j(n)a^k(n) = 0,$$

$$[X^j(n) - X^j(n-1)][X^k(n) - X^k(n-1)] = 0$$

if $j \neq k$, which means that these martingales are *orthogonal.* In the study of related problems in continuous time the analogue of this fact plays an important rôle. ∎

When the action rule β is used, the measures $P_i^\beta := P_{e_i}^\beta$, corresponding to $\xi := e_i$, $i = 1, \dots, N$, play a special rôle. The measure P_i^β is concentrated on the set $\{\theta_i = 1\}$ and corresponds to the measure P_ξ^β under the i^{th} hypothesis. From (2.2)–(2.4) it follows that the representation

$$P_\xi^\beta := \sum_{i=1}^{N} \xi_i P_i^\beta \tag{2.5}$$

holds for the measure P_ξ^β. Further, for given ξ and β the corresponding expectations will be denoted by E_ξ^β and E_i^β and, in places where it is clear which ξ and β are being discussed, we will simply write E.

Let $\Delta X^j(n) := X^j(n) - X^j(n-1)$ for $n \geq 1$ and let

$$V_\nu^\beta(\xi) := E_\xi^\beta \sum_{j=1}^{m} X^j(\nu) = E_\xi^\beta \sum_{n=1}^{\nu} \sum_{j=1}^{m} \Delta X^j(n)$$

denote the expectation of the resulting number of successes for ν steps upon applying the action rule β with *a priori* distribution ξ. Then, according to (2.3) and (2.4), taking account of the fact that $X(0) := 0$, we have that

$$\begin{aligned} V_\nu^\beta(\xi) &= E_\xi^\beta \sum_{n=1}^{\nu} \sum_{i=1}^{N} \theta_i \sum_{j=1}^{m} \lambda_i^j a^j(n) \\ &= \sum_{n=1}^{\nu} E_\xi^\beta\, \theta\Lambda a^*(n) = \sum_{n=1}^{\nu} E_\xi^\beta\, \theta\Lambda\beta^*(n). \end{aligned} \tag{2.6}$$

We are interested in the problem of finding the action rule β which maximizes the resulting *expected number of successes* for ν steps. Accordingly, let the *value function* be defined by

$$V_\nu(\xi) := \sup_\beta V_\nu^\beta(\xi).$$

Denote by $\tilde{\Lambda}$ the matrix in which all m elements of the i^{th} row, $i = 1, \dots, N$, coincide with

$$\lambda_i := \max_{j=1,\dots,m} \lambda_i^j. \tag{2.7}$$

If we knew before the observations which hypothesis held (i.e. if the action rule would be understood to be not $\mathcal{F}_{n-1}$- but $\mathcal{F}^{\theta}_{n-1}$-measurable), then the expectation of the *maximum possible* number of successes for ν steps would obviously coincide with $\nu \sum_{i=1}^N \xi_i\lambda_i$. The latter value, using (2.2) and the equality $\sum_{j=1}^m a^j(n) := 1$, may be rewritten in the form

$$\begin{aligned} &\sum_{n=1}^{\nu} \sum_{i=1}^{N} \xi_i \sum_{j=1}^{m} \lambda_i a^j(n) \\ &\qquad = \sum_{n=1}^{\nu} \xi\tilde{\Lambda} a^*(n) = \sum_{n=1}^{\nu} E_\xi^\beta\, \theta\tilde{\Lambda} a^*(n)\,. \end{aligned}$$

Therefore the functional

$$W_\nu^\beta(\xi) := \nu \sum_{i=1}^{N} \xi_i\lambda_i - V_\nu^\beta(\xi) = \sum_{n=1}^{\nu} E_\xi^\beta \theta(\tilde{\Lambda} - \Lambda)a^*(n) \tag{2.8}$$

is naturally termed the *loss* for the time interval of length ν using the action rule β with *a priori* distribution ξ and the problem of *minimization* of $W_\nu^\beta(\xi)$ may be solved instead of the equivalent problem of *maximization* of the functional $V_\nu^\beta(\xi)$. Correspondingly,

$$W_\nu(\xi) := \inf_\beta W_\nu^\beta(\xi) \tag{2.9}$$

is called the *loss function* for time ν. Designating $W_\nu^\beta(e_i) =: W^\beta_{(i)\nu}$, relation (2.8), using (2.5), may be rewritten in the form

$$W_\nu^\beta(\xi) = \sum_{i=1}^{N} \xi_i \sum_{n=1}^{\nu} E_i^\beta(\lambda_i - \lambda_i^j)a^j(n) = \sum_{i=1}^{N} \xi_i W^\beta_{(i)\nu}\,. \tag{2.10}$$

Remark 2.2 We term an action rule the sequence $\beta := \{\beta(n)\}$ given above for all $n = 1, 2, \dots$. Naturally, for studying loss up to time ν, an *action rule* is understood as a sequence of random vectors $\beta(n)$ given for $n \le \nu$ and considered to be measurable with respect to $\mathcal{F}_\nu^\theta$. Further, considering the loss up to time ν we will not specially distinguish whether or not $\beta(n)$ is defined *only* for $n \le \nu$ or for *all* natural n. ■

Note that the right-hand side of formula (2.8), by nonnegativity of the expression under the expectation operation, also makes sense for $\nu = \infty$, and the value

$$W_\infty^\beta(\xi) := \sum_{n=1}^{\infty} E_\xi^\beta\, \theta(\tilde{\Lambda} - \Lambda)a^*(n) \tag{2.11}$$

corresponds to the loss over an *infinite* time interval, and is either finite or equals $+\infty$. By analogy with (2.9) we define

$$W_\infty(\xi) := \inf_\beta W_\infty^\beta(\xi). \tag{2.12}$$

From (2.8) and (2.9) it follows that $W_\nu(\xi)$ is monotonically increasing with respect to ν and $W_\nu(\xi) \le W_\infty(\xi)$.

The question of coincidence of $W(\xi) := \lim_{\nu\to\infty} W_\nu(\xi)$ with the value $W_\infty(\xi)$ and, in the case of finite $W_\infty(\xi)$, the question of existence and properties of the action rule realizing the infimum in (2.12) are of interest. The answers to these questions will be given in the following sections. The remainder of *this* section is devoted to another representation of our problem which is equivalent to the one given above.

Before we move on to this representation, we point out that in spite of the fact that in the sequel we will be mainly interested in the number of successes or the equivalent loss (i.e. the functionals $V_\nu^\beta(\xi)$ and $W_\nu^\beta(\xi)$ for $\nu \le \infty$), many of the general results given in the sequel hold for arbitrary additive functionals which, for $\nu \le \infty$, have the form

$$F_\nu^\beta(\xi) := \sum_{n=1}^{\nu} E_\xi^\beta\, f_n(\theta, h(n)) \tag{2.13}$$

(assuming that the corresponding expectations and sums are defined). From (2.8), the function W is obtained if in (2.13) we take $f_n :=$

$\theta(\tilde{\Lambda} - \Lambda)a^*(n)$. As before, let $\theta(\tilde{\Lambda} - \Lambda)a^*(n)$. As before, let

$$F_\nu(\xi) := \inf_\beta F_\nu^\beta(\xi), \qquad F(\xi) := \lim_{u\to\infty} F_\nu(\xi) \tag{2.14}$$

for $\nu \le \infty$, assuming that the latter limit exists. As for the function W, questions arise of the coincidence of $F_\infty(\xi)$ and $F(\xi)$ and of the existence of optimal action rules.

* * *

We now introduce the *a posteriori* probabilities of the hypotheses that will play a significant rôle in the investigation of the problems considered. In particular, it will be shown that we may transform the initial problem to a different representation in which the parameter θ is absent and the *a posteriori* probabilities are included in the state space of the system.

Let ξ be some fixed value of the *a priori* probabilities of hypotheses. For $n = 0, 1, \dots$, let

$$\xi^\beta(n) := E_\xi^\beta(\theta|\mathcal{F}_n).$$

This relation may be rewritten in coordinate form as

$$\xi_i^\beta(n) := E_\xi^\beta(\theta_i|\mathcal{F}_n) = P_\xi^\beta\{\theta_i = 1|\mathcal{F}_n\}, \qquad i = 1, \dots, N. \tag{2.15}$$

Since the σ-algebra $\mathcal{F}_n$ is generated by a finite number of mutually exclusive events, then for the corresponding events the conditional probabilities in (2.15) may be understood as ordinary conditional probabilities with respect to these events.

It will be shown that the sequence $\xi^\beta(n)$ may be chosen *independently* of β.

Let

$$p^j(\xi) := \sum_{i=1}^{N} \lambda_i^j \xi_i \tag{2.16}$$

and define the operators Γ^{1j}, Γ^{0j}, $j = 1, \dots, m$, acting on S^N by the following formulae:

$$\begin{aligned}
(\Gamma^{1j}\xi)_i &:= \begin{cases} \lambda_i^j \xi_i / p^j(\xi) & \text{if } p^j(\xi) \ne 0, \\ \xi_i & \text{if } p^j(\xi) = 0 \end{cases} \\
(\Gamma^{0j}\xi)_i &:= \begin{cases} (1 - \lambda_i^j)\xi_i / (1 - p^j(\xi)) & \text{if } p^j(\xi) \ne 1, \\ \xi_i & \text{if } p^j(\xi) = 1, \end{cases}
\end{aligned} \tag{2.17}$$

which are operators defining the change in *a posteriori* probabilities after applying action a^j and (subsequently) observing the event of a success (1) or a failure (0).

We may now introduce the stochastic process

$$\xi(0) = \xi, \qquad \xi(n) = \sum_{j=1}^{m} a^j(n)[\Delta X^j(n)\Gamma^{1j}\xi(n-1) + (1 - \Delta X^j(n))\Gamma^{0j}\xi(n-1)]. \tag{2.18}$$

Lemma 2.1 *For all β and $n \geq 0$ the equalities*

$$\xi^\beta(n) = \xi(n) = E_\xi^\beta(\theta|\tilde{\mathcal{F}}_{n+1}) \qquad (P_\xi^\beta \text{ a.s.})$$

hold (where a.s. *denotes* almost surely, *i.e. with probability* 1*).*

Proof. The equality $\xi^\beta(0) = \xi(0)$ (P_ξ^β a.s.) follows from (2.2). Suppose that $\xi^\beta(s) = \xi(s)$ (P_ξ^β a.s.) for all s, $0 \leq s \leq n-1$. As previously stated, for fixed n the σ-algebra $\mathcal{F}_n$ is generated by a finite number of mutually exclusive events on which $\xi^\beta(s)$ and $\xi(s)$ take constant values. Therefore, without loss of generality it may be assumed that the equality $\xi^\beta(s) = \xi(s)$ holds for $0 \leq s \leq n-1$ for *any* ω. We show that this equality also holds for $s = n$.

It is sufficient to check that if B is one of the mutually exclusive events generating $\mathcal{F}_n$ then $P_\xi^\beta(\theta_i = 1|B) = \xi_i(n)I\{B\}$. Here $I\{B\}$ denotes the *indicator* function of the set B, i.e. $I\{B\}(\omega) = 1$ or 0 according as $\omega \in B$ or not. Since $\mathcal{F}_n = \mathcal{F}_{n-1} \vee \sigma\{a(n), \Delta X(n)\}$, then there exists a j, $1 \leq j \leq m$, such that B may be represented either as

$$B = B' \cap \{a^j(n) = 1\} \cap \{\Delta X^j(n) = 0\}$$

or as

$$B = B' \cap \{a^j(n) = 1\} \cap \{\Delta X^j(n) = 1\}, \tag{2.19}$$

where B' is one of mutually exclusive events generating $\mathcal{F}_{n-1}$. Consider first the case where the representations (2.19) and $P_\xi^\beta\{B\} > 0$ hold, in which case it follows from (2.19) that $P_\xi^\beta\{B'\} > 0$.

In the remainder of the proof we will omit the indices ξ and β of the measure P_ξ^β and the argument n for the values $a^j(n)$ and $X^j(n)$.

By Bayes' theorem we have from (2.15) that

$$\xi_i^\beta(n)I\{B\} = P\{\theta_i = 1|B\}I\{B\} = \frac{P\{\theta_i = 1\}P\{B|\theta_i = 1\}}{\sum_{k=1}^m P\{\theta_k = 1\}P\{B|\theta_k = 1\}} I\{B\}. \tag{2.20}$$

From (2.19) and the elementary properties of conditional probabilities it follows that

$$P\{B|\theta_k = 1\} = P\{B'|\theta_k = 1\}P\{a^j = 1|B', \theta_k = 1\}P\{\Delta X^j = 1|a^j = 1, B', \theta_k = 1\}. \tag{2.21}$$

By (2.3), $I\{B'\}P\{a^j = 1|B', \theta_k = 1\} = \beta^j(n)I\{B'\}$, and by (2.21) and the fact that $P\{B\} > 0$, it follows that $\beta^j(n) > 0$ on B'. Further, from (2.4) we have that $P\{\Delta X^j = 1|a^j = 1,\ B',\ \theta_k = 1\} = \lambda_k^j$. Putting these values in (2.21) and then the result in (2.20), taking into account that

$$\begin{aligned} P\{B'|\theta_k = 1\}P\{\theta_k = 1\}I\{B'\} &= P\{\theta_k = 1|B'\} \cdot P\{B'\}I\{B'\} \\ &= \xi_k^\beta(n-1) \cdot P\{B'\}I\{B'\} \\ &= \xi_k(n-1) \cdot P\{B'\}I\{B'\}, \end{aligned}$$

and cancelling $\beta^j(n)P\{B'\}$ in the numerator and denominator, we have that

$$\xi_i^\beta(n)I\{B\} = \xi_i(n-1)\lambda_i^j \cdot I\{B\} / \sum_{k=1}^N \lambda_k^j \xi_k(n-1). \tag{2.22}$$

Thus on the set B, in accordance with (2.18), $\xi(n)I\{B\} = I\{B\}\Gamma^{1j}\xi(n-1)$, i.e. $\xi^\beta(n)I\{B\} = \xi(n)I\{B\}$. The case in which in (2.19) $\Delta X^j = 0$ is considered similarly, and if the set B is such that $P_\xi^\beta\{B\} = 0$, then $\xi^\beta(n)$ may be defined arbitrarily on this set and, in particular, using the right-hand side of formula (2.18). Therefore, the first equality of Lemma 2.1 is proven.

To prove the second equality of Lemma 2.1, it is sufficient to show that if B is one of the mutually exclusive events generating $\mathcal{F}_n$ and

$P\{B \cap \{a^j(n+1)=1\}\} > 0$, then $P\{\theta_i = 1 | B \cap \{a^j(n+1)=1\}\} = P\{\theta_i = 1|B\}$. This is true because using (2.3)

$$\begin{aligned} &P\{a^j(n+1)=1|B\}I(B) \\ &\qquad = P\{a^j(n+1)=1|B,\theta_i=1\}I(B) = \beta^j(n+1)I(B). \quad \blacksquare \end{aligned}$$

Let us show now that the expression for the functional (2.13) may be transformed so that it does not depend on θ. In fact, since θ_i, a^j, $\Delta X^j(n)$ have only values 0 or 1 then the function $f_n(\theta, h(n)) := f_n(\theta, h(n-1), a(n), \Delta X(n))$ may be represented as a function of ω as

$$\begin{aligned} &f_n(\theta, \cdot, a(n), \Delta X(n)) \\ &\quad = \sum_{i=1}^{N} \theta_i \sum_{j=1}^{m} a^j(n)[f_{in}^{1j}(\cdot)\Delta X^j(n) + f_{in}^{0j}(\cdot)(1-\Delta X^j(n))], \end{aligned}$$

where $f_{in}^{1j}(\cdot) := f_n(e_i^N, \cdot, e_j^m, e_j^m)$, $f_{in}^{0j} := f_n(e_i^N, \cdot, e_j^m, e_0^m)$. Converting to conditional expectation with respect to $\tilde{\mathcal{F}}_n^{\theta}$, and taking account of (2.4) leaves

$$\begin{aligned} &E[f_n(\theta, h(n))|\tilde{\mathcal{F}}_n^{\theta}] \\ &\quad = \sum_{i=1}^{N} \theta_i \sum_{j=1}^{m} a^j(n)[f_{in}^{1j}(h(n-1))\lambda_i^j + f_{in}^{0j}(h(n-1))(1-\lambda_i^j)]. \end{aligned}$$

Converting this expression to conditional expectation with respect to $\tilde{\mathcal{F}}_n$, and by the second equality of Lemma 2.1 for $n-1$, and defining

$$\begin{aligned} &q_n(\xi, \cdot, a(n)) \\ &\quad := \sum_{i=1}^{N} \xi_i \sum_{j=1}^{m} a^j(n)[\lambda_i^j f_{in}^{1j}(\cdot) + (1-\lambda_i^j) f_{in}^{0j}(\cdot)], \end{aligned} \tag{2.23}$$

we have that

$$\begin{aligned} F_\nu^\beta(\xi) &= \sum_{n=1}^{\nu} E_\xi^\beta\, f_n(\theta, h(n)) \\ &= \sum_{n=1}^{\nu} E_\xi^\beta\, q_n(\xi(n-1), h(n-1), a(n)). \end{aligned} \tag{2.24}$$

Here $q_n := \xi(\tilde{\Lambda} - \Lambda)a^*$ corresponds to the *loss*, so that

$$W_\nu^\beta(\xi) = \sum_{n=1}^{\nu} E_\xi^\beta\, \xi(n-1)(\tilde{\Lambda} - \Lambda)a^*(n)\,. \tag{2.25}$$

Expression (2.24) for the functional $F_\nu^\beta(\xi)$ allows us to obtain another representation of the initial problem in which the process $\xi(n)$ is present and the random value θ is absent.

Notice first of all that since the right-hand side of formula (2.3) is $\mathcal{F}_{n-1}$-measurable by the definition of an action rule, then from (2.3) it follows that

$$P_\xi^\beta\{a^j(n)=1|\mathcal{F}_{n-1}\} = \beta^j(n). \tag{2.26}$$

Further, taking conditional expectation with respect to $\tilde{\mathcal{F}}_n$ in (2.4), and using the results of Lemma 2.1 and expression (2.16), we obtain

$$P_\xi^\beta\{\Delta X^j(n)=1|\tilde{\mathcal{F}}_n\} = a^j(n)p^j(\xi(n-1)). \tag{2.27}$$

Relations (2.26) and (2.27), together with formula (2.18) defining $\xi(n)$, give a complete restriction of the measure P_ξ^β on $\mathcal{F}_\infty^\theta$ to $\mathcal{F}_\infty$. According to (2.24), to compute the functional (2.13) it is sufficient to know only this restriction.

Therefore, we obtain the following representation of the initial problem. Suppose given X and a on $(\Omega, \mathcal{F})$ satisfying Conditions (b) and (c), and the σ-algebras $\mathcal{F}_n$ and $\tilde{\mathcal{F}}_n$, $n \le \infty$, defined as in (2.1). The process $\xi(n)$ is defined for fixed $\xi \in S^N$ by formula (2.18). The vector ξ and the action rule β may be used to construct a measure on $\mathcal{F}_\infty$ (not on $\mathcal{F}_\infty^\theta$, as before) which is uniquely defined by relations (2.26) and (2.27) (this measure may be considered to be the restriction to $\mathcal{F}_\infty$ of the original measure P_ξ^β). It is required to choose an action rule to minimize the function (2.24).

In such a representation the problem formulated is represented as a problem of control of the process $(\xi(n), X(n))$, and from (2.18) and (2.26) and (2.27) it follows that

$$\begin{aligned} P_\xi^\beta\{\xi(n) &= \Gamma^{1j}\xi(n-1), \Delta X^j(n) = 1|\tilde{\mathcal{F}}_n, a^j(n)=1\} \\ &= p^j(\xi(n-1))a^j(n), \\ P_\xi^\beta\{\xi(n) &= \Gamma^{0j}\xi(n-1), \Delta X^j(n) = 0|\tilde{\mathcal{F}}_n, a^j(n)=1\} \\ &= [1-p^j(\xi(n-1))]a^j(n). \end{aligned} \tag{2.28}$$

These relations are interpreted in the following manner. If at time $(n-1)$ the *a posteriori* probabilities of hypotheses are equal to $\xi(n-1)$ and at time n the j^{th} device is observed, then with probability $p^j(\xi(n-1))$ a jump will be realized in the j^{th} coordinate and the *a posteriori* probabilities will become $\Gamma^{1j}\xi(n-1)$. With the complementary probability a jump will *not* occur and the *a posteriori* probabilities will become $\Gamma^{0j}\xi(n-1)$.

This conversion of the problem reflects the fact mentioned in §2.2 that for general problems of sequential control with incomplete information in Bayes' formulation the *a posteriori* probabilities of the values of unknown parameters together with the values of the original processes are sufficient statistics for the investigation of additive functionals (see §2.4).

In many cases it is possible to further reduce the problem to control of the process $\xi(n)$, *without* explicit knowledge of the process $X(n)$. To effect this we will assume that on $(\Omega,\mathcal{F})$ we are given only a sequence $a := \{a(n), n=1,2,\dots\}$, where $a(n)$ takes values in $\widehat{S}^m$, and a process $\xi(n)$ such that $\xi(0)=\xi$ and for $a^j(n)=1$ the process $\xi(n)$ may take either the value $\Gamma^{1j}\xi(n-1)$ or $\Gamma^{0j}\xi(n-1)$. We introduce the σ-algebras

$$\mathcal{F}^*_n := \sigma\{\xi(s), a(s),\ 1 \le s \le n\},$$

$$\widetilde{\mathcal{F}}^*_{n+1} := \mathcal{F}^*_n \vee \sigma\{a(n+1)\}, \qquad n \ge 0.$$

An action rule β^* is now defined as a sequence $\{\beta^*(n), n=1,2,\dots\}$, where $\beta^*(n)$ takes values in S^m and is $\mathcal{F}^*_{n-1}$-measurable for each $n \ge 1$. As before, each action rule β^* for a given value $\xi(0) := \xi$ corresponds to a measure $P^{\beta^*}_\xi$ on $\mathcal{F}^*_\infty$ satisfying the conditions

$$P^{\beta^*}_\xi\{a^j(n)=1 | \mathcal{F}^*_{n-1}\} = \beta^{*j}(n)$$

$$P^{\beta^*}_\xi\{\xi(n) = \Gamma^{1j}\xi(n-1) | \mathcal{F}^*_{n-1}, a^j(n)=1\} = p^j(\xi(n-1))a^j(n) \quad (2.29)$$

$$P^{\beta^*}_\xi\{\xi(n) = \Gamma^{0j}\xi(n-1) | \mathcal{F}^*_{n-1}, a^j(n)=1\} = (1-p^j(\xi(n-1)))a^j(n).$$

If the matrix Λ is such that it has no column with all elements coinciding, then from (2.17) it follows that $\Gamma^{1j} \neq \Gamma^{0j}$ for all $j = 1,\dots,m$. In this case, from (2.18) and the inclusion $\{a^j(s)=1\} \supseteq \{\Delta X^l(s)=0,\ l \neq j\}$ it follows that $\mathcal{F}^*_n = \mathcal{F}_n$ and $\widetilde{\mathcal{F}}^*_n = \widetilde{\mathcal{F}}_n$. Therefore $h(n-1)$ is expressed in terms of $a(1),\dots,a(n-1)$ and $\xi(0),\dots,\xi(n-1)$. From (2.24) it then follows that the minimization problem with respect to all possible action rules of the functional

$$F^{\beta^*}_\nu(\xi) = \sum_{n=1}^{\nu} E^{\beta^*}_\xi\, q_n(\xi(n-1), h(n-1), a(n)) \qquad (2.30)$$

is equivalent to the initial problem.

If the matrix Λ has at least one column with coinciding elements, then for the corresponding column $\Gamma^{1j}\xi = \xi = \Gamma^{0j}\xi$, and therefore $\mathcal{F}^*_n \subset \mathcal{F}_n$ and $\widetilde{\mathcal{F}}^*_n \subset \widetilde{\mathcal{F}}_n$, and the inclusion is strong. In this case the set of action rules in the new formulation will be smaller than in the previous formulation of the problem. Nevertheless, if $f_n(\theta, h(n))$ in (2.13) depends only on the control and the increment of the process at the previous moment of time, i.e. has the form $f_n(\theta, a(n), \Delta X(n))$ (so that q_n in (2.30) depends only on $\xi(n-1)$ and $a(n)$), then for each old action rule $\beta = \{\beta(n),\ n=1,2,\dots\}$ a new action rule $\beta^* = \{\beta^*(n),\ n=1,2,\dots\}$ may be found such that $F^{\beta^*}_\nu(\xi) = F^{\beta}_\nu(\xi)$. It is sufficient to use $\beta^*(n) := E^\beta_\xi\{\beta(n)|\mathcal{F}^*_{n-1}\}$. Here the measure $P^{\beta^*}_\xi$ is a restriction to $\mathcal{F}^*_\infty$ of the measure P^β_ξ given on $\mathcal{F}_\infty$. From this it follows that in this case it is sufficient to restrict oneself to action rules in their new interpretation for the solution of the original minimization problem and both problems are again equivalent (the general version of this statement is discussed in §2.2).

In some works the problems given above are immediately formulated in terms of control of *a posteriori* probabilities according to the formulae (2.24) and (2.29). This approach is often sufficient to obtain an answer. However, the important representation (2.5) is lost with this approach, and also the corresponding representation of the functional (see, for example, (2.10)). Such a representation for F^β_ν will be used below (see (2.57)) in the proof of Theorem 2.2.

We now show the connection of the problem formulated in this section with the general problem of sequential control with incomplete information.

2.2 General problems of sequential control with incomplete information

To begin we make a few comments with respect to notation. Capital script letters are used for spaces (possibly with indices), and elements are denoted by the corresponding lower-case letters (with the same indices). If $\mathcal{Y}$ is a measurable space then $\mathcal{P}(\mathcal{Y})$ is the set of all probability measures on $\mathcal{Y}$. The measurable structure on $\mathcal{Y}$ is given by the minimal σ-algebra with respect to which all integrals of bounded functions are measurable. Further, $\mathcal{Y}_{sn} := \prod_{r=s}^{n} \mathcal{Y}_r$, so that

$$y_{sn} := (y_s, \ldots, y_n), \qquad y_r \in \mathcal{Y}_r, \qquad r = s, \ldots, n.$$

The general ***sequential control problem with incomplete information*** in the ***discrete time*** case is given by the object

$$(\mathcal{X}_n, \mathcal{A}_{n+1}, \mathcal{H}_n, \widetilde{\mathcal{H}}_{n+1}, \Theta, \{p_n^\theta, q_{n+1}^\theta, \theta \in \Theta\},\ n = 0, 1, \ldots). \tag{2.31}$$

Its components are the following:-

(1) $\mathcal{X}_n$ is the ***state space*** of the system at time n.

(2) $\mathcal{A}_{n+1}$ is the ***control space*** at time n.

The spaces $\mathcal{X}_n$ and $\mathcal{A}_{n+1}$ are assumed to be measurable spaces.

For the controlled system the choice of control at time n may not be possible in the full space $\mathcal{A}_{n+1}$, but only in its ***subset*** generated by the history up to time n, i.e. by the ***trajectory*** $x_0, a_1, x_1, a_2, x_2, \ldots, a_n, x_n$, which we shall often simply express in the form $x_0 a_1 x_1 a_2 x_2 \ldots a_n x_n$ for convenience. Similarly, the set of ***admissible states*** at time $n+1$ depends on $x_0 a_1 \ldots x_n a_{n+1}$. To take into account such restrictions for $n \geq 0$ we introduce the sets $\mathcal{H}_n$ and $\widetilde{\mathcal{H}}_{n+1}$.

(3) $\mathcal{H}_n$ is a measurable subset of the product space $\mathcal{X}_0 \times \mathcal{A}_1 \times \cdots \times \mathcal{X}_n$; $\widetilde{\mathcal{H}}_{n+1}$ is a measurable subset of the space $\mathcal{H}_n \times \mathcal{A}_{n+1}$ defining the set of ***admissible histories***.

Suppose that $\mathcal{H}_0 := \mathcal{X}_0$, $\mathcal{H}_n \subset \widetilde{\mathcal{H}}_n \times \mathcal{X}_n$ and assume that for each $h_n \in \mathcal{H}_n$ the section of the set $\widetilde{\mathcal{H}}_{n+1}$ at the point h_n is not empty. The elements of such a section are called ***admissible controls*** for h_n. It is similarly assumed that for each $\tilde{h}_n \in \widetilde{\mathcal{H}}_n$ the section at the point $\tilde{h}_n$ of $\mathcal{H}_{n+1}$ is nonempty, i.e. the set of admissible transitions for $\tilde{h}_n$ is nonempty.

Sometimes in the infinite product $\mathcal{X}_0 \times \mathcal{A}_1 \times \mathcal{X}_1 \times \ldots$ it is convenient to consider the set of all admissible *infinite* histories which is defined as

$$\mathcal{H}_\infty := \bigcap_{n=1}^{\infty} (\mathcal{H}_n \times \mathcal{A}_{n+1} \times \mathcal{X}_{n+1} \times \ldots). \tag{2.32}$$

Then it is sometimes also convenient to consider functions given on $\mathcal{H}_n, \widetilde{\mathcal{H}}_{n+1}$ as functions given on $\mathcal{H}_\infty$ and depending only on the appropriate coordinates.

Further, we will sometimes omit the time index n for h_n, $\tilde{h}_n$, a_n, x_n.

It is assumed that each admissible trajectory $\tilde{h} \in \widetilde{\mathcal{H}}_{n+1}$ at time n, i.e. each admissible history $h \in \mathcal{H}_n$, together with a control $a \in \mathcal{A}_{n+1}$ defines a probability distribution on the set of admissible states of the process (i.e. on the corresponding section of the set $\mathcal{H}_{n+1}$ at the point $\tilde{h}$ at time $n+1$).

(4) Θ is the ***parameter set***.

(5) $p_0^\theta(\cdot) := p_0^\theta$ is a probability measure on $\mathcal{H}_0$ depending on θ and defining the initial state of the system; $p_{n+1}^\theta(\cdot|\tilde{h}) := p_{n+1}^\theta$ is a probability measure on $\mathcal{X}_{n+1}$, defining a probability distribution at time $n+1$ which depends on the history $\tilde{h} \in \widetilde{\mathcal{H}}_{n+1}$ and the value of the parameter θ.

We assume that Θ is a measurable space, that $p_{n+1}^\theta(\cdot|\tilde{h})$ is concentrated on the admissible set of states, and that p_n^θ, $n \geq 0$ is a ***transition function*** from $\widetilde{H}_n \times \Theta$ into $\mathcal{X}_n$, where $\widetilde{\mathcal{H}}_0 \times \Theta := \Theta$. (By a transition function from $\mathcal{Y}$ into $\mathcal{X}$ we mean a function which depends on y and on the measurable subsets B of the space $\mathcal{X}$ such that for each fixed y in $\mathcal{Y}$ it defines a probability measure on $\mathcal{X}$ and for each fixed measurable B in $\mathcal{X}$ it depends measurably on y.

For fixed θ the sequence $p^\theta := \{p_n^\theta, n = 0, 1, \ldots\}$ is called a ***strategy of nature***.