Type vectors

The *type vector* of a partition of n, or of a set partition of $[n]$, is an expression of the form $1^{k_1}2^{k_2}\cdots n^{k_n}$ where k_i is the number of parts of size i.

For example $3+3+1$ is a partition of 7 with type vector $1^1 2^0 3^2 4^0 5^0 6^0 7^0$, and $\{\{2\},\{4\},\{1,3\}\}$ is a set partition of $[4]$ with type vector $1^2 2^1 3^0 4^0$.

Obviously, a type vector is valid if and only if

$$k_1 \cdot 1 + k_2 \cdot 2 + \cdots + k_n \cdot n = n$$

is satisfied. Subject to this condition *there is exactly one partition of n for each given type vector.*

Theorem 1 *The number of set partitions of $[n]$ with type vector $1^{k_1}2^{k_2}\cdots n^{k_n}$ is*

$$\frac{n!}{(1!)^{k_1}(2!)^{k_2}\cdots(n!)^{k_n}k_1!k_2!\cdots k_n!}.$$

Proof: Let us list the elements of the set partition in an increasing order of part size, and let us list the elements of each part in any order. Separate the parts with vertical bars, thus obtaining a *code* for the set partition. For example, for the set partition $\{\{2\},\{4\},\{1,3\}\}$ we obtain the following four valid codes:

$$2|4|1,3 \quad 2|4|3,1 \quad 4|2|1,3 \quad \text{and} \quad 4|2|3,1.$$

For any fixed type vector there are $n!$ possible valid codes: we need to specify only the order of the elements, the positions of the vertical bars are determined by the type vector. We only need to check that each set partition has exactly $(1!)^{k_1}(2!)^{k_2}\cdots(n!)^{k_n}k_1!k_2!\cdots k_n!$ valid codes. Indeed, for each $i \in [n]$, there is $i!$ ways to permute the elements within a part of size i (hence we have a factor of $(i!)^{k_i}$) and there are $k_i!$ ways to permute the parts of size i. $\Diamond$