# Fibonacci-type sequences (easy approach)

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A Fibonacci-type sequence $a_0, a_1, \ldots$ is given by a recurrence

$$a_{n+2} + ba_{n+1} + ca_n = 0 \tag{1}$$

and by the initial values for $a_0$ and $a_1$. For example, for the Fibonacci numbers we have $b = c = -1$ and $a_0 = a_1 = 1$. We want to find a closed formula for such a sequence.

The key idea is to find two geometric sequences $1, q_1, q_1^2, \ldots$ and $1, q_2, q_2^2, \ldots$, satisfying the recurrence (1) and to express the sequence $a_0, a_1, \ldots$ as a linear combination of these geometric sequences. That is, we want to find numbers $\alpha$ and $\beta$ satisfying

$$a_n = \alpha q_1^n + \beta q_2^n \quad \text{for all } n \geq 0. \tag{2}$$

A geometric sequence $1, q, q^2, \ldots$ satisfies the recurrence (1) exactly when the number $q$ is a solution of the *characteristic equation*

$$q^2 + aq + b = 0.$$

We may use our key idea without any modification when the characteristic equation has two distinct roots. (Note that these roots may be complex!) The exceptional case when the characteristic equation has a double root, will be handled afterward. Once we have found the roots $q_1$ and $q_2$, we may find the coefficients $\alpha$ and $\beta$ in (2) by substituting $n = 0$ and $n = 1$, respectively, and solving the resulting system of linear equations

$$\left.\begin{array}{rcl} a_0 & = & \alpha + \beta \\ a_1 & = & \alpha q_1 + \beta q_2 \end{array}\right\} \tag{3}$$

This has a unique solution for $\alpha$ and $\beta$, since the determinant of the coefficient matrix is

$$\det \begin{pmatrix} 1 & 1 \\ q_1 & q_2 \end{pmatrix} = q_2 - q_1 \neq 0.$$

For example, for the Fibonacci numbers, we obtain the characteristic equation

$$q^2 - q - 1 = 0$$

whose solution is

$$q_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad q_2 = \frac{1-\sqrt{5}}{2}.$$

The system of equations (3) takes the form

$$\left.\begin{array}{rcl} 1 &=& \alpha+\beta \\ 1 &=& \alpha\dfrac{1+\sqrt{5}}{2}+\beta\dfrac{1-\sqrt{5}}{2} \end{array}\right\}$$

whose solution is $\alpha=\dfrac{1}{\sqrt{5}}\cdot\dfrac{1+\sqrt{5}}{2}$ and $\beta=\dfrac{1}{\sqrt{5}}\cdot\dfrac{\sqrt{5}-1}{2}$. Thus we obtain

$$a_n=\frac{1}{\sqrt{5}}\cdot\frac{1+\sqrt{5}}{2}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^n+\frac{1}{\sqrt{5}}\cdot\frac{\sqrt{5}-1}{2}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

which may be simplified to

$$a_n=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right).$$

In the *exceptional case* when the characteristic equation has a double root $q_1$, the same $q_1$ is a double root of the polynomial $q^{n+3}+aq^{n+2}+bq^{n+1}$ for each $n\geq 0$. Thus $q_1$ is still a root of the derivative $(n+3)q^{n+2}+(n+2)aq^{n+1}+(n+1)bq^n$. In other words, the sequence $1,2q_1,3q_1^2,\ldots,(n+1)q_1^n,\ldots$ also satisfies the same recurrence. From this we may subtract the geometric sequence $1,q_1,q_1^2,\ldots$ and obtain that the sequence $0,q_1,2q_1^2,\ldots,nq_1^n,\ldots$ satisfies the same recurrence. Thus we may look for the solution in the form

$$a_n=\alpha q_1^n+\beta n q_1^n \quad \text{for all } n\geq 0. \tag{4}$$

instead of (2). Substituting $n=0$ and $n=1$ into this equation yields the system of equations

$$\left.\begin{array}{rcl} a_0 &=& \alpha \\ a_1 &=& (\alpha+\beta)q_1 \end{array}\right\} \tag{5}$$

which has the unique solution $\alpha=a_0$ and $\beta=a_1q_1^{-1}-a_0$, *whenever* $q_1\neq 0$. Finally the case when $q_1=0$ is a double root, the characteristic equation is $q^2=0$ and we have $a_{n+2}=0$ for $n\geq 0$.

An example for the double root situation would be $a_{n+2}-6a_{n+1}+9a_n=0$ where the characteristic equation is $q^2-6q+9=0$, with double root $q_1=3$. Assuming $a_0=0$ and $a_1=1$, after solving (5) we get $\alpha=0$ and $\beta=3^{-1}$. Thus $a_n=0+3^{-1}n3^n=n3^{n-1}$ for $n\geq 0$.