Spherical trigonometry

1 The spherical Pythagorean theorem

Proposition 1.1 *On a sphere of radius R, any right triangle $\triangle ABC$ with $\angle C$ being the right angle satisfies*

$$\cos(c/R) = \cos(a/R)\cos(b/R). \tag{1}$$

Proof: After replacing a/R, b/R and c/R with a, b, and c we may assume $R = 1$. We rotate the triangle in such a way that $\overrightarrow{OC} = (0, 0, 1)$, A is in the xz plane and B is in the yz-plane, see Figure 1. A rotation around O in the xz plane by $b = \angle AOC$ takes C into A, thus we have

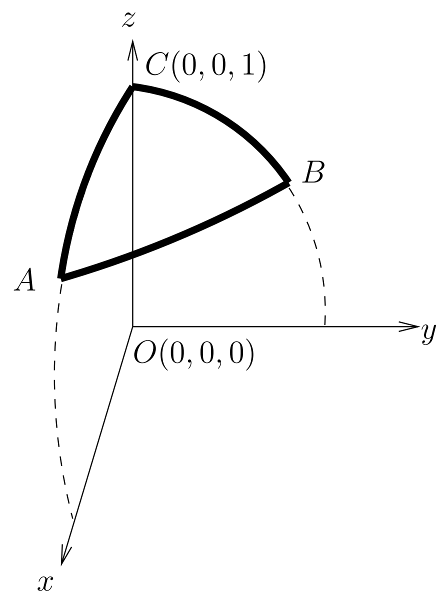

Figure 1: A spherical right triangle

$$\overrightarrow{OA} = (\sin(b), 0, \cos(b)).$$

Similarly, a rotation around O in the yz-plane by $a = \angle BOC$ takes C into B, thus we have

$$\overrightarrow{OB} = (0, \sin(a), \cos(a)).$$

The difference of the two vectors is

$$\overrightarrow{AB} = (-\sin(b), \sin(a), \cos(a) - \cos(b)).$$

Hence the length of AB satisfies

$$\begin{aligned} |AB|^2 &= \sin^2(b) + \sin^2(a) + (\cos(a) - \cos(b))^2 = \sin^2(b) + \sin^2(a) + \cos^2(a) + \cos^2(b) - 2\cos(a)\cos(b) \\ &= 2 - 2\cos(a)\cos(b). \end{aligned}$$

Applying the law of cosines to the isosceles right triangle $OAB_\triangle$ we get

$$|AB|^2 = |OA|^2 + |OB|^2 - 2 \cdot |OA| \cdot |OB| \cdot \cos(c), \quad \text{that is,}$$

$$2 - 2\cos(a)\cos(b) = 2 - 2\cos(c).$$

After subtracting 2 on both sides, and dividing both sides by (-2) we obtain the stated equation. $\Diamond$

To prove the rest of the formulas of spherical trigonometry, we need to show the following.

Proposition 1.2 *Any spherical right triangle $\triangle ABC$ with $\angle C$ being the right angle satisfies*

$$\sin(A) = \frac{\sin\left(\frac{a}{R}\right)}{\sin\left(\frac{c}{R}\right)} \quad \textit{and} \tag{2}$$

$$\cos(A) = \frac{\tan\left(\frac{b}{R}\right)}{\tan\left(\frac{c}{R}\right)}. \tag{3}$$

Proof: We continue assuming $R = 1$ and using Figure 1. The angle A is between $\overrightarrow{OA} \times \overrightarrow{OB}$ and $\overrightarrow{OA} \times \overrightarrow{OC}$. Here

$$\overrightarrow{OA} \times \overrightarrow{OB} = (-\cos(b)\sin(a), -\sin(b)\cos(a), \sin(a)\sin(b)) \quad \text{and} \quad \overrightarrow{OA} \times \overrightarrow{OC} = (0, -\sin(b), 0).$$

The length of $\overrightarrow{OA} \times \overrightarrow{OB}$ is $\left|\overrightarrow{OA}\right| \cdot \left|\overrightarrow{OB}\right| \cdot \sin(c) = \sin(c)$, the length of $\overrightarrow{OA} \times \overrightarrow{OC}$ is $\sin(b)$.

To prove (2) we use the fact that

$$\left|(\overrightarrow{OA} \times \overrightarrow{OB}) \times (\overrightarrow{OA} \times \overrightarrow{OC})\right| = \left|\overrightarrow{OA} \times \overrightarrow{OB}\right| \cdot \left|\overrightarrow{OA} \times \overrightarrow{OC}\right| \cdot \sin(A). \tag{4}$$

Since

$$(-\cos(b)\sin(a), -\sin(b)\cos(a), \sin(a)\sin(b)) \times (0, -\sin(b), 0) = (\sin(a)\sin^2(b), 0, \sin(b)\cos(b)\sin(a))$$

the left hand side of (4) is

$$\sqrt{\sin^2(a)\sin^4(b) + \sin^2(b)\cos^2(b)\sin^2(a)} = \sin(b)\sin(a)\sqrt{\sin^2(b) + \cos^2(b)} = \sin(b)\sin(a).$$

The right hand side of (4) is is $\sin(b)\sin(c)\sin(A)$. Thus we have

$$\sin(b)\sin(a) = \sin(b)\sin(c)\sin(A),$$

yielding (2).

To prove (3) we use the fact that

$$(\overrightarrow{OA} \times \overrightarrow{OB}) \cdot (\overrightarrow{OA} \times \overrightarrow{OC}) = \left|\overrightarrow{OA} \times \overrightarrow{OB}\right| \cdot \left|\overrightarrow{OA} \times \overrightarrow{OC}\right| \cdot \cos(A). \tag{5}$$

The left hand side is $\sin^2(b)\cos(a)$, the right hand side is $\sin(b)\sin(c)\cos(A)$. Thus we obtain

$$\sin^2(b)\cos(a) = \sin(b)\sin(c)\cos(A),$$

yielding

$$\cos(A) = \frac{\sin(b)\cos(a)}{\sin(c)} = \frac{\tan(b)}{\tan(c)} \cdot \frac{\cos(a)\cos(b)}{\cos(c)}.$$

Equation (3) now follows from Proposition 1.1. $\Diamond$

2 General spherical triangles

To prove the spherical laws of sines and cosines, we will use the Figure 2.

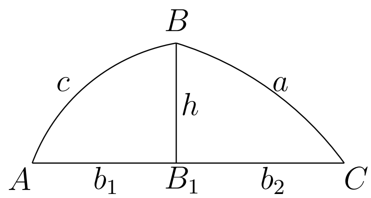

Figure 2: A general spherical triangle

Theorem 2.1 (Spherical law of sines) *Any spherical triangle satisfies*

$$\frac{\sin(A)}{\sin(a/R)} = \frac{\sin(B)}{\sin(b/R)} = \frac{\sin(C)}{\sin(c/R)}.$$

Proof: Applying (2) to the right triangle ABB_1 yields

$$\sin(A) = \frac{\sin(h/R)}{\sin(c/R)}.$$

This equation allows us to express $\sin(h/R)$ as follows:

$$\sin(h/R) = \sin(A)\sin(c/R).$$

Similarly, applying (2) to the right triangle CBB_1 allows us to write

$$\sin(h/R) = \sin(C)\sin(a/R).$$

Therefore we have

$$\sin(A/R)\sin(c/R) = \sin(C)\sin(a/R),$$

since both sides equal $\sin(h/R)$. Dividing both sides by $\sin(a/R)\sin(c/R)$ yields

$$\frac{\sin(A)}{\sin(a/R)} = \frac{\sin(C)}{\sin(c/R)}.$$

The equality

$$\frac{\sin(A)}{\sin(a/R)} = \frac{\sin(B)}{\sin(b/R)}$$

may be shown in a completely similar fashion. $\Diamond$

Theorem 2.2 (Spherical law of cosines) *Any spherical triangle satisfies*

$$\cos(a/R) = \cos(b/R)\cos(c/R) + \sin(b/R)\sin(c/R)\cos(A).$$

Proof: Applying (1) to the right triangle $\triangle BB_1C$ yields

$$\cos(a/R) = \cos(b_2/R)\cos(h/R)$$

Let us replace b_2 with $b - b_1$ in the above equation. After applying the formula $\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$ we obtain

$$\cos(a/R) = \cos(b/R)\cos(b_1/R)\cos(h/R) + \sin(b/R)\sin(b_1/R)\cos(h/R).$$

Applying (1) to the right triangle $\triangle BB_1A$ we may replace both occurrences of $\cos(h/R)$ above with $\cos(c/R)/\cos(b_1/R)$ and obtain

$$\cos(a/R) = \cos(b/R)\cos(c/R) + \sin(b/R)\sin(b_1/R)\frac{\cos(c/R)}{\cos(b_1/R)}, \quad \text{that is,}$$

$$\cos(a/R) = \cos(b/R)\cos(c/R) + \sin(b/R)\sin(c/R)\frac{\tan(b_1/R)}{\tan(c/R)}.$$

Finally, (3) applied to the right triangle $\triangle BB_1A$ allows replacing $\tan(b_1/R)/\tan(c/R)$ with $\cos(A)$. $\Diamond$

To obtain the spherical law of cosines for angles, we may apply the preceding theorem to the *polar triangle* of the triangle $\triangle ABC$. This one has sides $a' = (\pi - A)R$, $b' = (\pi - B)R$ and $c' = (\pi - C)R$ and angles $A' = \pi - a/R$, $B' = \pi - b/R$ and $C' = \pi - c/R$. The spherical law of cosines for the triangle $\triangle A'B'C'$ states

$$\cos(a'/R) = \cos(b'/R)\cos(c'/R) + \sin(b'/R)\sin(c'/R)\cos(A'), \quad \text{that is,}$$

$$\cos(\pi - A) = \cos(\pi - B)\cos(\pi - C) + \sin(\pi - B)\sin(\pi - C)\cos(\pi - a/R).$$

Using $\cos(\pi - x) = -\cos(x)$ and $\sin(\pi - x) = \sin(x)$, after multiplying both sides by (-1) we obtain

$$\cos(A) = -\cos(B)\cos(C) + \sin(B)\sin(C)\cos(a/R). \tag{6}$$

References

[1] D. Royster, "Non-Euclidean Geometry and a Little on How We Got There," Lecture notes, May 7, 2012.